A DIOPHANTINE EQUATION

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1. Introduction. Partial integer solutions of the diophantine equation

$$X^2 - Y^2 = Z^3 - W^3, \tag{1.1}$$

some involving integer parameters, but all far from complete, have been given. In contrast, the more difficult equation in which the left member is X^2+Y^2 (obviously it is immaterial whether the right member is Z^3-W^3 or Z^3+W^3) was completely solved in integers by Rosenthall [8] by operating in a certain (Dirichlet) biquadratic domain. For (1.1) the rational domain suffices throughout. The solution goes through on the strictly elementary level because Dickson [2, 4, 5] recast his first proof [1] of the theorem of which Lemma 2 below is an instance so as to obviate ideals, and indeed algebraic numbers, entirely. This was possible because multiplication of ideals is equivalent to composition of certain (composable) forms. Each of the three lemmas is a very special case of a corresponding general theorem having numerous applications to diophantine analysis. None has yet been fully exploited.

The structure of the complete integer solution of (1.1) is best seen by leaving it in unreduced form. Each of X, Y is of degree 9, and each of Z, W of degree 6, in the 9 independent integer parameters a, b, c, f, g, h, k, n, p; the integers $\alpha, \beta, \gamma, \zeta, \eta$ are defined by

$$\alpha^2 = \beta^2 = \gamma^2 = 1; \quad (\zeta, \eta) = (\gamma, 4\gamma), \quad (2\gamma, 2\gamma), \quad (4\gamma, \gamma). \tag{1.2}$$

The solution falls into two parts, (A), (B).

$$\begin{aligned} 8X &= j[\eta x(t^2 + 3v^2) + \zeta w(r^2 + 3s^2)], \\ 8Y &= j[\eta x(t^2 + 3v^2) - \zeta w(r^2 + 3s^2)], \\ 2Z &= j[\alpha(rv + st) + \beta(rt - 3sv)], \\ 2W &= j[\alpha(rv + st) - \beta(rt - 3sv)], \end{aligned} \tag{A}$$

in which j, x, w, r, s, t, v are integers ranging over all solutions $(j, \cdots, v)$ of

$$xw = \beta j(rt - 3sv). \tag{1.3}$$

All solutions of (1.3) are given by

$$\begin{aligned} j = \beta np, \quad x = p(ag + bh), \quad w = ckn, \quad r = ak, \\ s = kb, \quad t = bf + cg, \quad v = (af - ch)/3, \end{aligned} \tag{1.4}$$

where a, f, c, h are such that

$$af \equiv ch \bmod 3. \tag{1.5}$$

The values (1.4) of $j, \cdots, v$ substituted into (A) give the first part of the complete integer solution of (1.1). We shall omit the details of solving (1.5) explicitly and assigning the forms modulo 8 of the parameters.

$$
\begin{aligned}
4X &= j[\eta x(t^2 + tv + v^2) + \zeta w(r^2 - rs + s^2)], \\
4Y &= j[\eta x(t^2 + tv + v^2) - \zeta w(r^2 - rs + s^2)], \\
2Z &= j[\alpha(rv + st) + \beta(2rt - 2sv - st + vr)], \\
2W &= j[\alpha(rv + st) - \beta(2rt - 2sv - st + vr)],
\end{aligned} \tag{B}
$$

in which $(j, \cdots, v)$ ranges over all sets of integer solutions of

$$xw = \beta j(2rt - 2sv - st + vr). \tag{1.6}$$

All solutions of (1.6) are given by

$$
\begin{aligned}
&j = \beta np, \quad x = -p(ag + bh), \quad w = -kcn, \quad r = bf + cg, \\
&s = -af + ch, \quad t = k(2a + b)/3, \quad v = -k(a + 2b)/3,
\end{aligned} \tag{1.7}
$$

where k, a, b are such that

$$k(a - b) \equiv 0 \bmod 3. \tag{1.8}$$

It is readily verified that (A), (B) actually are solutions. Their completeness follows from the lemmas stated next.

2. Three lemmas. In the following lemmas the complete integer solutions of the respective equations are as indicated.

LEMMA 1. [6]

$$RS = TV:$$

$$R = rt, \quad S = sv, \quad T = rv, \quad V = st,$$

r, s, t, v independent integer parameters, t, v, coprime if desired.

LEMMA 2. [4, 5]

$$R^2 + 3S^2 = TV:$$

$$
\begin{aligned}
&R = \beta j(rt - 3sv), && S = j(rv + st), \\
&T = j(r^2 + 3s^2), && V = j(t^2 + 3v^2);
\end{aligned} \tag{A'}
$$

$$
\begin{aligned}
&R = \beta j(2rt - 2sv - st + vr), && S = j(rv + st), \\
&T = j(2r^2 - 2rs + 2s^2), && V = j(2t^2 + 2tv + 2v^2),
\end{aligned} \tag{B'}
$$

j, r, s, t, v independent integer parameters.

LEMMA 3. [7]

$$X_1Y_1 + X_2Y_2 + X_3Y_3 = 0:$$

$$
\begin{aligned}
X_1 &= kx_1, \quad & Y_1 &= x_2y_1 + x_3y_2, \\
X_2 &= kx_2, \quad & Y_2 &= -x_1y_1 + x_3y_3, \\
X_3 &= kx_3, \quad & Y_3 &= -x_1y_2 - x_2y_3,
\end{aligned}
$$

the small letters independent integer parameters.

3. Verification. By the change of notation $X+Y=X'$, $X-Y=Y'$, (1.1) becomes

$$X'Y' = (Z - W)(Z^2 + ZW + W^2). \tag{3.1}$$

By application of Lemma 1,

$$\begin{aligned} X' &= xz, & Z - W &= xw \\ Y' &= yw, & Z^2 + ZW + W^2 &= yz; \end{aligned} \tag{3.2}$$

whence, by elimination of W,

$$3Z^2 - 3xwZ + x^2w^2 - yz = 0, \tag{3.3}$$

to be solved in integers Z, x, y, z, w. The Z-discriminant of (3.3) is the integer $12yz-3x^2w^2$. This must be the square, $(3u)^2$, of an integer multiple u of 3:

$$(xw)^2 + 3u^2 = 4yz.$$

Referring to (1.2), we write $y'=\eta y$, $z'=\zeta z$. Then

$$(xw)^2 + 3u^2 = y'z'. \tag{3.4}$$

Thus far, with Z, W from (3.2), (3.3), and $2X=X'+Y'$, $2Y=X'-Y'$, we have

$$\begin{aligned} 8X = \eta xz' + \zeta wy', \qquad 8Y = \eta xz' - \zeta wy', \qquad 2Z = \alpha u + xw, \\ 2W = \alpha u - xw, \end{aligned} \tag{3.5}$$

where x, w, u, y', z' are to be determined from (3.4).

Application of Lemma 2 to (3.4) and substitution of the values of x, w, u, y', z' thus determined into (3.5) give the two parts (A), (B) of the solution, where now j, x, w, r, s, t, v are to be found from (1.3) for (A), and from (1.6) for (B).

Application of Lemma 1 to (1.3) gives

$$x = pR, \qquad w = Qn, \qquad \beta j = pn, \qquad rt - 3sv = QR,$$

and by application of Lemma 3 to the last of these written as

$$r\cdot t + s\cdot(-3sv) + Q\cdot(-R) = 0$$

we get

$$\begin{aligned} r &= ka, & t &= bf + cg, \\ s &= kb, & 3v &= af - ch, \\ Q &= kc, & R &= ag + bh. \end{aligned}$$

Collecting results we have (1.4), (1.5). This disposes of (A).

From Lemma 1 applied to (1.6), $x=pR$, $w=Qn$, $\beta j=pn$,

$$(2t + v)\cdot r - (t + 2v)\cdot s - Q\cdot R = 0;$$

whence, by Lemma 3,

$$2t + v = ka, \qquad r = bf + cg,$$
$$t + 2v = -kb, \qquad s = -af + ch,$$
$$Q = -kc, \qquad R = -ag - bh.$$

Collecting results we have (1.7), (1.8), disposing of (B).

References

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2. L. E. Dickson, Integral solutions of $x^3 - my^3 = zw$, ibid., vol. 29 (1923), pp. 464–467.
3. L. E. Dickson, All integral solutions of $ax^3 + bxy + cz^3 = w_1 w_2 \cdots w_n$, ibid, vol. 32 (1936), pp. 644–648.
4. L. E. Dickson, Introduction to the theory of numbers 1929, Chap. VI. (The discrepancy between Theorem 69, p. 91 and Exercise 1, p. 93, is rectified in [5], p. 186, Exercise 3. Apparently the sign $\pm$ should be prefixed to the value of x in the first part of the solution stated on p. 186.)
5. L. E. Dickson, Modern elementary theory of numbers, 1939, 3d. impression, 1947, Ch. IX.
6. Stated as Problem 1, p. 195, in [5].
7. Alternative form in [5], Section 84. The more symmetrical form of the solution used here is a special case of F. Giudice's solution; see Th. Skolem, Diophantische Gleichungen, 1938.
8. E. Rosenthall, Diophantine equations reducible in biquadratic fields, Duke Math. Jr., vol. 10 (1943), pp. 463–470.

ON THE MONGE POINT OF THE TETRAHEDRON*

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1. Introduction. In recent years some properties, possibly new, have been added to this chapter of the geometry of the tetrahedron [1]. We shall mention some of them after having generalized the definition of the Monge point in order to call attention to various spheres associated with it. Many of the properties of these spheres seem to have remained unpublished.

2. History. In a memoir entitled, *Sur la pyramide triangulaire*, Monge proved that the planes drawn through the midpoints of the edges of a tetrahedron perpendicular to the opposite edges meet at a point M, and that M is the midpoint of the segment joining the centroid and the circumcenter [2]. In this memoir the hyperboloid of center M, determined by the altitudes, is not mentioned. Its fundamental properties seem to have been cited first by Steiner [3]. According to Chasles, Monge and Hachette had found the rectilinear generators of these second degree surfaces analytically. He adds, "For a long time the only demonstration of this property of the hyperboloid was the initial analytical proof. While a student at the École Polytechnique I gave a purely geometric discussion, which became a part of the course at Polytechnique" [4].

* Translated from the French by Col. W. E. Byrne.

3. Generalization. The properties proved by Monge are particular cases of the following:

THEOREM I. *If the planes drawn through the midpoints of the edges of a tetrahedron $T \equiv ABCD$ perpendicular to the corresponding (or opposite) edges of a tetrahedron $T' \equiv A'B'C'D'$ intersect in a point Q', the planes drawn through the midpoints of the edges of T' perpendicular to the corresponding (or opposite) edges of T intersect in a point Q.*

Let A_1, B_1, C_1, A_1', B_1', C_1' and a_1, b_1, c_1, a_1', b_1', c_1' be the midpoints of the edges BC, CA, AB, DA, DB, DC and $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ of T and T'. If the planes drawn through the points A_1, B_1, C_1 perpendicular to the edges $B'C'$, $C'A'$, $A'B'$ of T' intersect in a point Q', these planes are coaxal and the triangles $A_1B_1C_1$ (or ABC) and $A'B'C'$ are said to be skew orthological.* The planes through a_1, b_1, c_1 perpendicular to the sides BC, CA, AB of triangle ABC have in common a line d_d. For the same reason the planes perpendicular to the sides AB, DA, DB of triangle DAB drawn through c_1, a_1', b_1' have a common line d_c. The lines d_d and d_c, both in the plane through c_1 perpendicular to AB, meet in a point Q, common to the five planes considered. The plane drawn through c_1' perpendicular to DC is coaxal with the planes drawn through a_1 perpendicular to BC and through b_1' perpendicular to DB; hence this sixth plane also contains Q, thus proving the theorem.

If we replace the corresponding edges by the opposite edges, the planes in question meet at the points Q_1 and Q_1', symmetrical to Q and Q' with respect to the centroids of T' and T. The points T and T' are orthological with respect to the midpoints of the corresponding (or opposite) edges. The points Q, and Q' (or Q_1, and Q_1') are their respective centers of orthology [6]. If T and T' coincide, Q and Q' coincide with the circumcenter O of T; Q_1 and Q_1' coincide with the point symmetric to O with respect to the centroid G of T. We find thus the construction indicated by Monge of the point M which bears his name and observe at the same time the fundamental property of this point.

4. Notations. Let the edges of $T \equiv ABCD$ be denoted by $BC=a$, $DA=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$. Also let us designate by (O, R) the circumsphere; G the centroid; G_a, G_b, G_c, G_d the centroids of the faces BCD, CDA, DAB, ABC; G_a', G_b', G_c', G_d' the midpoints of the medians $AG_a=M_a$, $BG_b=M_b$, $CG_c=M_c$, $DG_d=M_d$; O_a, O_b, O_c, O_d the circumcenters of the faces opposite the vertices A, B, C, D; A_1', B_1', C_1', D_1' the points of intersection of the altitudes AA', BB', CC', DD' with the circumsphere.

5. Metric relations. Let O' be the orthogonal projection of O on the altitude AA' and let O_i, G_i', M_i', $(i=a, b, c, d)$, be the orthogonal projections on the planes BCD, CDA, DAB, ABC of O, G, M. We have (in magnitude and sign)

* If the planes drawn through the vertices of a triangle ABC perpendicular to the corresponding sides of a triangle $A'B'C'$ are coaxal, then the planes drawn through the vertices of $A'B'C'$ perpendicular to the corresponding sides of ABC are coaxal. [5].

$$OO_a = \frac{AA' + A'A_1'}{2} + A_1'A' = \frac{AA' + A_1'A'}{2}, \cdots$$

$$GG_a'' = \frac{OO_a + MM_a'}{2} = \frac{AA'}{4}, \cdots.$$

From the above relations it follows that

$$MM_a' = \frac{AA'}{2} + O_aO = \frac{A'A_1'}{2},$$

and

$$MM_b' = \frac{B'B_1'}{2}, \qquad MM_c' = \frac{C'C_1'}{2}, \qquad MM_d' = \frac{D'D_1'}{2}.$$

Hence we have

THEOREM II. *In a tetrahedron T the necessary and sufficient condition that the Monge point M be interior or exterior to T is that the points A and A_1', B and B_1', C and C_1', D and D_1' be separated, respectively, by the planes of the faces BCD, CDA, DAB, ABC or that one of these pairs of points be on the same side of the plane of the corresponding face.*

COROLLARY. *For the point M to be on the surface of the tetrahedron, in the plane of a face, or on an edge, or at a vertex, it is necessary and sufficient that the foot of the altitude drawn from the opposite vertex, the feet of the two altitudes drawn from the vertices opposite the faces which intersect along the edge in question, the feet of the three altitudes drawn from the three other vertices, respectively, be on the circumsphere.*

We recall the formulas:

$$\overline{MO}^2 = 4R^2 - \sum \frac{a^2 + a'^2}{4}, \tag{1}$$

$$M_a^2 = \frac{a'^2 + b^2 + c^2}{3} - \frac{a^2 + b'^2 + c'^2}{9}, \cdots \tag{2}$$

$$\rho_a^2 = \frac{a^2 + b'^2 + c'^2}{18}, \cdots \tag{3}$$

where ρ_a is the radius of the director circle of the inscribed Steiner ellipse of triangle BCD. By applying the theorem on medians to the triangles AOM, BOM, COM, DOM and AMG_a, BMG_b, CMG_c, DMG_d, then using Stewart's theorem for the four last named triangles to calculate MG_i, we obtain [7],

$$\overline{MA}^2 = R^2 + \frac{a'^2 + b^2 + c^2 - a^2 - b'^2 - c'^2}{4}, \cdots \tag{4}$$

$$(5) \qquad \overline{MG}_a^2 = R^2 - \frac{3(a'^2 + b^2 + c^2) + a^2 + b'^2 + c'^2}{36}, \cdots$$

$$(6) \qquad \overline{MG}_a'^2 = R^2 - \frac{a^2 + b'^2 + c'^2}{9} = \overline{OG}_a^2, \cdots$$

which will be used later.

6. Spheres associated with the Monge point. (a) The Longchamps sphere of the medial tetrahedron t of T.

THEOREM III. *In a tetrahedron $T \equiv ABCD$, twice the sum of the squares of the edges issued from a vertex diminished by the sum of the squares of the edges of the opposite face equals six times the power (P_i) of that vertex with respect to the Longchamps sphere of the medial tetrahedron $t \equiv G_aG_bG_cG_d$.*

In calling attention to the Longchamps sphere of a tetrahedron T for the first time, we established that its center M_1 coincides with the symmetric of the Monge point M with respect to the circumcenter O of T, and that its radius 3ρ is given by [8],

$$(7) \qquad 9\rho^2 = 9R^2 - 3\sum \frac{a^2 + a'^2}{4}.$$

The Longchamps sphere of t is the transform of the sphere $(M_1, 3\rho)$ by the homothety $(G, -1/3)$; its center coincides with M.* The theorem may be proved by using (4) and (7) to show that

$$(P_a) = \overline{AM}^2 - \rho^2 = \frac{2(a'^2 + b^2 + c^2) - a^2 - b'^2 - c'^2}{6}, \cdots.$$

COROLLARY. *In a tetrahedron T the power (P_i') of a vertex with respect to the Longchamps sphere $(M_1, 3\rho)$ equals half the sum of the squares of the edges of the opposite face. The sum of the powers of the four vertices equals the sum of the squares of the edges.*

The median theorem applied to triangle AMM_1 gives

$$\overline{AM}_1^2 = 9R^2 - \frac{3(a'^2 + b^2 + c^2) + a^2 + b'^2 + c'^2}{4}$$

and

$$(P_a') = \overline{AM}_1^2 - 9\rho^2 = \frac{a^2 + b'^2 + c'^2}{2}, \cdots$$

so that

$$\sum (P_i') = \sum (a^2 + a'^2).$$

* The sphere (M, ρ) coincides with the quasi-polar sphere of T of N. A. Court. [9]. Several of its properties given by N. A. Court are identical with those contained in our memoirs of 1932 and 1937 [8].

THEOREM IV. *In a tetrahedron $T \equiv ABCD$ the Monge point M coincides with the radical center of: the spheres described on the medians as diameters, the spheres which admit as great circles the director circles of the inscribed Steiner ellipses of the faces, the Longchamps spheres of the medial tetrahedrons of $T_a \equiv MBCD$, $T_b \equiv MCDA$, $T_c \equiv MDAB$, $T_d \equiv MABC$. These twelve spheres are orthogonal to the Longchamps sphere of the medial tetrahedron t.*

From formulas (1) to (7) we have

$$\overline{MG'_a}^2 - \frac{\overline{AG_a}^2}{4} = R^2 - \sum \frac{a^2 + a'^2}{12} = \frac{\overline{MO}^2 - R^2}{3} = \overline{MG_a}^2 - \rho_a^2.$$

Furthermore, by using the formula for (P_a) we obtain the power (P_M) of M with respect to the Longchamps where (M'_a) of the medial tetrahedron of T_a,

$$(P_M) = \frac{2(\overline{MB}^2 + \overline{MC}^2 + \overline{MD}^2) - (\overline{BC}^2 + \overline{BD}^2 + \overline{CD}^2)}{6}$$

$$= R^2 - \sum \frac{a^2 + a'^2}{12}.$$

Hence the point M is the radical center of the twelve spheres in question and its power with respect to these spheres is

$$R^2 - \sum \frac{a^2 + a'^2}{12} = \frac{\overline{MO}^2 - R^2}{3} = \rho^2;$$

they are all orthogonal to the Longchamps sphere (M, ρ) of t.

COROLLARY. *In a tetrahedron T the spheres $(G'_i, M_i/2)$ described on the medians as diameters intersect the spheres (G_i, ρ_i) traced on the director circles of the inscribed Steiner ellipses of the faces in four circles located on the director sphere (G) of the inscribed Steiner ellipsoid of T* [10].

COROLLARY. *The Longchamps sphere (M, ρ) of the medial tetrahedron t is orthogonal to the sphere (G) mentioned above* [8].

THEOREM V. *The thirteen spheres $(G'_i, M_i/2)$, (G_i, ρ_i), (M'_i) and (G) are orthogonal to (M, ρ).*

(b) Orthocentric sphere. By analogy with the orthocentric circle of a triangle (Tucker) let us agree to call the sphere (L) of center L described on MG a diameter the orthocentric sphere of T [11].

THEOREM VI. *Eight times the power of a vertex of T with respect to (L) equals twice the sum of the squares of the edges issued from this vertex less the sum of the squares of the edges of the opposite face.*

The powers (P_i) of the vertices A, B, C, D of T with respect to (L) are equal to

$$\frac{4\overline{AL}^2 - \overline{GM}^2}{4}, \quad \frac{4\overline{BL}^2 - \overline{GM}^2}{4}, \quad \frac{4\overline{CL}^2 - \overline{GM}^2}{4}, \quad \frac{4\overline{DL}^2 - \overline{GM}^2}{4}.$$

By application of the median theorem to triangles AGM, BGM, CGM, CGM we have, using formulas (1) to (6),

$$(P_a) = \frac{2(a'^2 + b^2 + c^2) - a^2 - b'^2 - c'^2}{8}, \quad \cdots \tag{8}$$

thus proving the theorem.

COROLLARY. *In a tetrahedron T the sum of the powers of the vertices with respect to (L) equals the sum of the square of the distances from the centroid G to the four vertices.*

By adding relations (8) we find

$$\sum (P_i) = \sum \frac{a^2 + a'^2}{4} = \overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2.$$

THEOREM VII. *In a tetrahedron $T \equiv ABCD$ the Monge point M is the radical center of the orthocentric spheres (L_a), (L_b), (L_c), (L_d) of tetrahedrons T_a, T_b, T_c, T_d.*

By virtue of (8) the power $P(M, L_a)$ of M with respect to (L_a) may be written

$$P(M, L_a) = \frac{2(\overline{MB}^2 + \overline{MC}^2 + \overline{MD}^2) - \overline{BC}^2 - \overline{CD}^2 - \overline{DB}^2}{8},$$

or

$$P(M, L_a) = \frac{[3R^2 - \sum (a^2 + a'^2)/4]}{4} = \frac{\overline{MO}^2 - R^2}{4} = \sigma^2. \tag{9}$$

Likewise,

$$P(M, L_b) = P(M, L_c) = P(M, L_d) = \sigma^2. \tag{10}$$

COROLLARY. *The diameter of the sphere (M, σ) orthogonal to the orthocentric spheres of T_i is equal to the radius of the director sphere of the Longchamps sphere of the medial tetrahedron t.*

We have

$$4\sigma^2 = \overline{MO}^2 - R^2 = 3R^2 - \sum \frac{a^2 + a'^2}{4} = 3\rho^2.$$

As a more general proposition, we have

THEOREM VIIa. *The radical center of spheres* (ω_i), $(i=a, b, c, d)$, *such that the powers of the vertices of* T_i *with respect to* (ω_i) *are* k *times twice the sum of the squares of the edges issued from these vertices less the sum of the squares of the edges of the opposite faces, is the Monge point* M *of* T, k *being positive or negative.*

By formula (4) we find for the power of M with respect to (ω_d),

$$k[2(\overline{MA}^2 + \overline{MB}^2 + \overline{MC}^2) - \overline{BC}^2 - \overline{CA}^2 - \overline{AB}^2] = k\left[\frac{12R^2 - \sum (a^2 + a'^2)}{2}\right] = 2k(\overline{MO}^2 - R^2).$$

As we shall see later, for $k=1/6$ the spheres (ω_i) are the Longchamps spheres of the medial tetrahedrons t_i of T_i. Theorems VII and VII a correspond to $k=1/8$. For $k=1/9$, the (ω_i) are the twelve-point spheres of T_i. From this we obtain the following proposition.

COROLLARY. *The Monge point* M *of* T *is the radical center of the twelve-point spheres of the tetrahedrons* T_i.

THEOREM VIII. *In a tetrahedron* T *the Monge point is the radical center of the spheres described on* Ag_a, Bg_b, Cg_c, Dg_d *as diameters, where* g_i *is the centroid of* T_i.

The twelve-point sphere $(\omega, R/3)$ of T (the circumsphere of the medial tetrahedron t of T) is the homothetic transform $(G, -1/3)$ of the circumsphere (O, R) of T. The sphere $(\omega, R/3)$ is also the homothetic transform $(M, 1/3)$ of (O, R). Consequently, the lines MG_a, MG_b, MG_c, MG_d meet (O, R) at the points A_2, B_2, C_2, D_2, diametrically opposite the vertices A, B, C, D, and also at the points A_3, B_3, C_3, D_3. Hence

$$MA_2 = 3MG_a = 4Mg_a.$$

The point A_3 is on the sphere described on Ag_a as a diameter. The power of M with respect to this sphere is

$$Mg_a \cdot MA_3 = \frac{MA_2 \cdot MA_3}{4} = \frac{\overline{MO}^2 - R^2}{4}, \quad \cdots \tag{11}$$

thus proving the theorem.

COROLLARY. *That the spheres described on* Ag_a, Bg_b, Cg_c, Dg_d *as diameters are orthogonal to* (M, σ) *follows from* (9), (10), *and* (11).

THEOREM IX. *In an orthocentric group of five tetrahedrons* $ABCDH$ *the orthocenter of any one of the orthocentric tetrahedrons* $T \equiv ABCD$, $T_a \equiv HBCD$, $T_b \equiv HCDA$, $T_c \equiv HDAB$, $T_d \equiv HABC$ *coincides with the radical center of the orthocentric spheres of the four other tetrahedrons.*

For this configuration the Monge points of tetrahedrons T and T_i coincide with the orthocenter H and the vertices A, B, C, D of the orthocentric tetrahedron T.

COROLLARY. *The radical planes of each of the orthocentric spheres of tetrahedrons T_i associated with the orthocentric sphere of T coincide with the planes of the faces of T, and the six radical axes of the orthocentric spheres of a pair of tetrahedrons T_i associated with the orthocentric sphere of T coincide with the edges of T.*

(c) Other spheres.

THEOREM X. *In a tetrahedron $T \equiv ABCD$ the spheres described on the director circles of the inscribed Steiner ellipses of the triangles BCD, CDA, DAB, ABC as great circles are orthogonal to the spheres described on MA, MB, MC, MD as diameters.*

The power of the centroid G_a of the face BCD with respect to the sphere (A_1) of center A_1 described on MA as a diameter has for its value

$$P(G_a) = \frac{4\overline{G_aA_1}^2 - \overline{MA}^2}{4} = \frac{\overline{AG_a}^2 + \overline{MG_a}^2 - \overline{MA}^2}{2}.$$

By formulas (1) to (6) the above expression may be reduced to

$$P(G_a) = \frac{a^2 + b'^2 + c'^2}{18} = \rho_a^2.$$

Analogous formulas hold for $P(G_b)$, $P(G_c)$ and $P(G_d)$, thus verifying the proposition.

COROLLARY. *In the medial tetrahedron t the vertices G_a, G_b, G_c, G_d are situated in the radical planes of the Longchamps sphere (M, ρ) associated with each of the spheres (A_1), (B_1), (C_1), (D_1).*

THEOREM XI. *In a tetrahedron T the circumsphere intersects the spheres $(G_a, \rho_a\sqrt{2})$ in great circles.*

From (3) and (6) it follows that

$$R^2 - \overline{OG_a}^2 = \frac{a^2 + b'^2 + c'^2}{9} = 2\rho_a^2, \cdots.$$

7. Coaxal pencils of spheres. The spheres (G_a'), (g_a'), (A_1) described on AG_a, Ag_a, AM as diameters have the points A and A_3 in common; AA_3 is their radical axis. Since the sphere (G_a, ρ_a) is orthogonal to (A_1) and to the Longchamps sphere (M, ρ) of the medial tetrahedron t, the radical plane of (A_1) and (M, ρ) perpendicular to A_1M, passes through G_a and intersects AA_3 at the radical center A_4 of (A_1), (M, ρ), (G_a'), (g_a'), (O, R). A_4M meets AG_a at right angles at A_5, since M is the orthocenter of the triangle AG_aA_4. Hence A_5 is on the orthocentric sphere (L),

$$A_4A_3 \cdot A_4A = A_4M \cdot A_4A_5,$$

the spheres (O, R), (L), (M, ρ), whose centers O, L, M are collinear, belong to a coaxial pencil (F) of spheres. Their common radical plane (π), perpendicular to OG, contains the point A_4 as well as the analogous points B_4, C_4, D_4. Furthermore, since the Monge point M has the same power with respect to the spheres (G_a') and (G_a, ρ_a) by Theorem IV, the radical plane of these spheres, perpendicular to $G_a'G_a$, contains the line A_4A_5. The line A_4A_5 is situated also in the radical plane of the spheres (G_a, ρ_a) and (G, σ). Hence the sphere (G, σ) is a fourth sphere of the coaxal pencil (F). If the radical plane (π) of (G, σ) and (O, R) intersects OG at K, we have

$$-\sigma^2 - (\overline{GO}^2 - R^2) = 2\overline{GO} \cdot \overline{GK}.$$

Since

$$\sigma^2 = \sum \frac{a^2 + a'^2}{48}, \text{ [13]}$$

and

$$\overline{GO}^2 = R^2 - \sum \frac{a^2 + a'^2}{16},$$

we find

$$\overline{GK} = \sum \frac{a^2 + a'^2}{48GO}.$$

The point ω, the center of the circumsphere $(\omega, R/3)$ of the medial tetrahedron t (twelve-point sphere of T), is on OG, and

$$\frac{G\omega}{GO} = -\frac{1}{3}$$

$$K\omega = KG - \frac{GO}{3}$$

$$KO = KG + GO.$$

As $\overline{KG}^2 - \sigma^2 = \overline{K\omega}^2 - R^2/9$, K has the same power with respect to (G, σ) and $(\omega, R/3)$. We conclude that $(\omega, R/3)$ belongs also to the coaxal pencil (F).

THEOREM XII. *The great circles cut from the spheres* (G_a, ρ_a), (G_b, ρ_b), (G_c, ρ_c), (G_d, ρ_d) *by the diametral planes of these spheres perpendicular to* MG_a, MG_b, MG_c, MG_d *are on a sphere* (m, r) *which belongs to the coaxal pencil* (F).

If m is the Monge point of t we observe that $mG_a = MA/3$, $mG_b = MB/3$, $mG_c = MC/3$, $mG_d = MD/3$. Formulas (3) and (5) show that

$$\overline{mG_a}^2 + \rho_a^2 = \frac{[R^2 + \sum (a^2 + a'^2)/4]}{9} = \overline{mG_b}^2 + \rho_b^2 = \cdots = r^2.$$

The common centroid G of T and t is at an algebraic distance d_g from the radical plane of (m, r) and (G, σ) such that

$$\overline{Gm}^2 - r^2 + \sigma^2 = 2\overline{Gm} \cdot d_g = \frac{2\overline{GO} \cdot d_g}{3}.$$

Hence $d_g = \sum(a^2+a'^2)/48\overline{OG} = \overline{GK}$; so (m, r) belongs to (F).

COROLLARY. *In an orthocentric tetrahedron T the director circles of the inscribed Steiner ellipses of the triangular faces of T are on the same sphere, whose center is the orthocenter of t.*

THEOREM XIII. *For a tetrahedron T the circumsphere, the twelve-point sphere, the orthocentric sphere, the director sphere of the inscribed Steiner ellipsoid, the Longchamps sphere of the medial tetrahedron t* [13] *and the sphere (m, r) belong to the same coaxal pencil (F).*

Note 1. The homothetic transform $(G, -3)$ of this configuration is another coaxal pencil of spheres to which we have called attention [13], adding $(M, 3r)$ as the transform of (m, r).

Note 2. This theorem generalizes and completes a known proposition relative to a triangle [14].

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A NOTE ON PURE RECURRENCE RELATIONS

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1. Introduction. The pure recurrence relations of many of the classical polynomials, for example, the Laguerre [5], and in particular those of the hypergeometric type, such as the Jacobi [5], Legendre [5], and so on, are readily obtained by the method exhibited in this paper. Many of the recently studied polynomials, as Rice's $H_n(\xi, p, v)$, [4] and [3], Bateman's $J_n^{u,v}$, [1] and [3], and Bateman's $Z_n(t)$ [1], lend themselves to this treatment. It is particularly useful for polynomials which do not form an orthogonal set, and for which there are not available the extremely simple methods of the theory of orthogonal polynomials.

2. Pure recurrence relation for Bateman's $Z_n(t)$. To illustrate we use the polynomial

$$Z_n(t) = {}_2F_2(-n, n+1; 1, 1; t) = \sum_{r=0}^{\infty} \frac{(-n)_r(n+1)_r t^r}{(r!)^3},$$

in which $(a)_r = a(a+1)\cdots(a+r-1)$; $(a)_0 = 1$. To simplify the exposition we let

$$Z_n(t) = \sum_{r=0}^{\infty} \beta_r$$

where, of course,

$$\beta_r = \frac{(-n)_r(n+1)_r t^r}{(r!)^3}.$$

We list a sequence of the $Z_k(t)$ and $tZ_k(t)$ $(k = n, n-1, \cdots)$ in which for clarity we exhibit each in both its explicit form and also in a form involving β_r as defined above. Thus

$$Z_n(t) = \sum_{r=0}^{\infty} \frac{(-n)_r\,(n+1)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \beta_r$$

$$Z_{n-1}(t) = \sum_{r=0}^{\infty} \frac{(-n+1)_r(n)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \frac{n-r}{n+r}\beta_r$$

$$tZ_{n-1}(t) = \sum_{r=1}^{\infty} \frac{(-n+1)_{r-1}(n)_{r-1} t^r}{(r-1)!^3} = \sum_{r=0}^{\infty} \frac{-r^3}{(n+r)(n-1+r)}\beta_r$$

$$Z_{n-2}(t) = \sum_{r=0}^{\infty} \frac{(-n+2)_r(n-1)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \frac{(n-r)(n-1-r)}{(n+r)(n-1+r)}\beta_r$$

$$tZ_{n-2}(t) = \sum_{r=1}^{\infty} \frac{(-n+2)_{r-1}(n-1)_{r-1} t^r}{(r-1)!^3} = \sum_{r=0}^{\infty} \frac{-r^3(n-r)}{(n+r)(n-1+r)(n-2+r)}\beta_r$$

$$Z_{n-3}(t) = \sum_{r=0}^{\infty} \frac{(-n+3)_r(n-2)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \frac{(n-r)(n-1-r)(n-2-r)}{(n+r)(n-1+r)(n-2+r)}\beta_r$$

If the coefficients of β_r in the above series are written with a least common denominator, and the numerators are polynomials of degree at most four in r, then a linear combination of the six series would have as numerator of its coefficient of β_r a polynomial of degree four in r. Such a linear combination would leave five undetermined constants with which to make the five coefficients in the fourth degree polynomial vanish. This suggests that for $n \geqq 3$ there exists a linear recurrence relation of the form

$$Z_n(t) + (A + Bt)Z_{n-1}(t) + (C + Dt)Z_{n-2}(t) + EZ_{n-3}(t) = 0, \tag{1}$$

in which A, B, C, D, and E are rational functions in n and are independent of t. To determine the coefficients in (1) we replace the several $Z_k(t)$ and $tZ_k(t)$ by their respective forms where the factor β_r occurs explicitly under the summation sign. Equating coefficients of β_r, we have for $r \geqq 0$

$$\begin{aligned} 1 &+ \frac{A(n-r)}{(n+r)} - \frac{Br^3}{(n+r)(n-1+r)} + \frac{C(n-r)(n-1-r)}{(n+r)(n-1+r)} \\ &- \frac{Dr^3(n-r)}{(n+r)(n-1+r)(n-2+r)} + \frac{E(n-r)(n-1-r)(n-2-r)}{(n+r)(n-1+r)(n-2+r)} = 0. \end{aligned} \tag{2}$$

Clearing the above expression of fractions gives the following identity in r:

$$\begin{aligned} &(r+n)(r+n-1)(r+n-2) + A(-r+n)(r+n-1)(r+n-2) \\ &\quad - Br^3(r+n-2) + C(-r+n)(-r+n-1)(r+n-2) \\ &\quad - Dr^3(-r+n) + E(-r+n)(-r+n-1)(-r+n-2) = 0. \end{aligned} \tag{3}$$

From this identity we can readily determine the coefficients in (1). Substituting the values thus obtained and clearing the result of fractions, we get, for $n \geqq 3$, the four-term recurrence relation:

$$\begin{aligned} &n^2(2n-3)Z_n - (2n-1)[3n^2 - 6n + 2 - 2(2n-3)t]Z_{n-1} \\ &\quad - (2n-3)[3n^2 - 6n + 2 + 2(2n-1)t]Z_{n-2} - (2n-1)(n-2)^2 Z_{n-3} = 0. \end{aligned} \tag{4}$$

3. A generalized hypergeometric polynomial with one parameter. We use a special case of a set of generalized hypergeometric polynomials [2], namely,

$$f_n(a; -; x) = {}_3F_2(-n, n+1, a; \tfrac{1}{2}, 1; x) = \sum_{r=0}^{\infty} \frac{(-n)_r(n+1)_r(a)_r x^r}{(\frac{1}{2})_r r!r!} \tag{5}$$

(a is independent of n and does not equal one or one-half) as a second example to depict the method in a more concise form. In the array below, the quantity

following the colon is the factor by which

$$\frac{(-n)_r(n+1)_r(a)_r x^r}{(\frac{1}{2})_r r!r!}$$

must be multiplied under the summation sign to give the respective expressions summed for each of the f_k and xf_k $(k=n, n-1, \cdots)$ where f_k is an abbreviation for $f_k(a; -; x)$. Then

$$\begin{aligned}
f_n &: 1 \\
f_{n-1} &: \frac{(-r+n)}{(r+n)} \\
xf_{n-1} &: \frac{-r^2(r-\frac{1}{2})}{(r+n)(r+n-1)(r+a-1)} \\
f_{n-2} &: \frac{(-r+n)(-r+n-1)}{(r+n)(r+n-1)} \\
xf_{n-2} &: \frac{-r^2(r-\frac{1}{2})(-r+n)}{(r+n)(r+n-1)(r+n-2)(r+a-1)} \\
f_{n-3} &: \frac{(-r+n)(-r+n-1)(-r+n-2)}{(r+n)(r+n-1)(r+n-2)}.
\end{aligned}$$

It is easily seen that when the above fractions are written with a least common denominator, neither it nor any numerator exceeds degree four in r. Hence, a linear combination of these fractions, with five constants to be determined, equated to zero gives an identity in r from which we can readily find the coefficients of a pure recurrence relation involving four of the f_k. For $n \geqq 3$

$$\begin{aligned}
(6)\qquad & nf_n - [(3n-2) - 4(n-1+a)x]f_{n-1} \\
& + [(3n-4) - 4(n-1-a)x]f_{n-2} - (n-2)f_{n-3} = 0.
\end{aligned}$$

It is worth noting that for the polynomial

$$f_n(-; -; x) = \sum_{r=0}^{\infty} \frac{(-n)_r(n+1)_r x^r}{(\frac{1}{2})_r r!r!} = \lim_{a\to\infty} f_n\left(a; -; \frac{x}{a}\right)$$

the recurrence relation

$$(7)\qquad nf_n - (3n-2-4x)f_{n-1} + (3n-4+4x)f_{n-2} - (n-2)f_{n-3} = 0$$

can be obtained from (6) by a limiting process. The reader may also derive (7) by using the method of this note.

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THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS THE SUM OF A PRIME AND A k-FREE INTEGER

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1. Introduction. T. Estermann* has discussed the representation of an integer as the sum of a square and a square-free number. Making use of a somewhat similar method, I shall establish the following result.

THEOREM 1. *Let k be any integer greater than 1, and H any positive number. Then every sufficiently large integer n can be represented as the sum of a prime and a k-free integer.*** *For $n\to\infty$ the number $T(n)$ of such representations is given by*

$$T(n) = \prod_{p\nmid n}\left(1 - \frac{1}{p^{k-1}(p-1)}\right)\operatorname{Li} n + O\left(\frac{n}{\log^H n}\right),$$

where

$$\operatorname{Li} n = \int_2^n \frac{du}{\log u},$$

and the O-constant depends at most on k and H.

2. Notation. Our notation is as follows. The symbols $\mu(n)$, $\phi(n)$ denote, as usual, the functions of Möbius and Euler, and $\mu_k(n)$ is defined as 1 or 0 according as n is or is not k-free.

The letter x denotes a certain function of n (to be fixed later) which tends to infinity with n; other small letters denote positive integers, and p is reserved for primes.

The highest common factor of a and b is denoted by (a, b).

The O-notation refers to the passage $n\to\infty$, and O-constants depend at most on k and H.

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* Einige Sätze über quadratfreie Zahlen, Math Annalen, vol. 105, 1931, pp. 654–662, §1.

** An integer is called k-free if it is not divisible by the kth power of any prime.

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3. Proof of Theorem 1. Since

$$\mu_k(m) = \sum_{a^k b = m} \mu(a),$$

we have

$$\begin{aligned} T(n) &= \sum_{m+p=n} \mu_k(m) = \sum_{a^k b + p = n} \mu(a) \\ &= \sum_{a^k b + p = n,\, a \leqq x} \mu(a) + \sum_{a^k b + p = n,\, a > x} \mu(a) \\ &= \sum_1 + \sum_2, \end{aligned}$$

say. To evaluate $\sum_1$ we use the well known result* that, for $(d, q) = 1$,

$$\sum_{p < n,\, p \equiv d \pmod q} 1 = \frac{1}{\phi(q)} \operatorname{Li} n + O\left(\frac{n}{\log^{2H} n}\right).$$

Hence we have

$$\begin{aligned} \sum_1 &= \sum_{a \leqq x} \mu(a) \sum_{p < n,\, p \equiv n \,(\mathrm{mod}\, a^k)} 1 \\ &= \sum_{a \leqq x,\,(a,n)=1} \mu(a) \left\{\frac{1}{\phi(a^k)} \operatorname{Li} n + O\left(\frac{n}{\log^{2H} n}\right)\right\} + O(x) \\ &= \operatorname{Li} n \sum_{(a,n)=1} \frac{\mu(a)}{\phi(a^k)} + O\left(\frac{n}{\log n} \sum_{a > x} \frac{1}{\phi(a^k)}\right) + O\left(\frac{nx}{\log^{2H} n}\right) \quad + O(x) \\ &= \prod_{p \nmid n} \left(1 - \frac{1}{\phi(p^k)}\right) \operatorname{Li} n + O\left(\frac{n}{\log n} \sum_{a > x} \frac{\log \log a}{a^k}\right) + O\left(\frac{nx}{\log^{2H} n}\right) + O(x) \\ &= \prod_{p \nmid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \operatorname{Li} n + O\left(\frac{n \log \log x}{x \log n}\right) + O\left(\frac{nx}{\log^{2H} n}\right) + O(x). \end{aligned}$$

Furthermore

$$\left|\sum_2\right| \leqq \sum_{a^k b + n = p,\, a > x} 1 \leqq \sum_{a^k b < n,\, a > x} 1 = O\left(\frac{n}{x^{k-1}}\right) = O\left(\frac{n}{x}\right).$$

Thus we have

$$\begin{aligned} T(n) = \prod_{p \nmid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \operatorname{Li} n + O\left(\frac{n \log \log x}{x \log n}\right) \\ + O\left(\frac{nx}{\log^{2H} n}\right) + O(x) + O\left(\frac{n}{x}\right), \end{aligned}$$

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* See, e.g., J. G. van der Corput, Sur l'hypothèse de Goldbach pour presque tous les nombres pairs, *Acta Arith.*, vol. 2, 1937, pp. 266–290, Footnote 4.

and Theorem 1 follows on putting $x = \log^H n$.

4. Another result. In his paper referred to previously, Estermann also obtained an asymptotic formula for the number of square-free integers not exceeding n and having the form a^2+l, where l is a given non-zero integer. Correspondingly, we have the following result.

THEOREM 2. *Let l be any non-zero integer, k any integer greater than 1, and H any positive number. Let, moreover, $U(n)$ denote the number of k-free integers $m \leqq n$, having the form $m = p+l$. Then, as $n \to \infty$,*

$$U(n) = \prod_{p \nmid l} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \operatorname{Li} n + O\left(\frac{n}{\log^H n}\right),$$

where the O-constant depends at most on k, l, and H.

The proof of this result is very similar to that of Theorem 1, and may be left to the reader.

MATHEMATICS

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Mathematics is one component of any plan for liberal education. Mother of all the sciences, it is a builder of the imagination, a weaver of patterns of sheer thought, an intuitive dreamer, a poet. The study of mathematics cannot be replaced by any other activity that will train and develop man's purely logical faculties to the same level of rationality. Through countless dimensions, riding high the winds of intellectual adventure and filled with the zest of discovery, the mathematician tracks the heavens for harmony and eternal verity. There is not wholly unexpected surprise, but surprise nevertheless, that mathematics has direct application to the physical world about us. For mathematics, in a wilderness of tragedy and change, is a creature of the mind, born to the cry of humanity in search of an invariant reality, immutable in substance, unalterable with time. Mathematics is an infinity of flexibles forcing pure thought into a cosmos. It is an arc of austerity cutting realms of reason with geodesic grandeur. Mathematics is crystallized clarity, precision personified, beauty distilled and rigorously sublimated. The life of the spirit is a life of thought; the ideal of thought is truth; everlasting truth is the goal of mathematics.

MATHEMATICAL NOTES

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A SIMPLE PROOF THAT, FOR ODD $p>1$, ARC COS $1/p$ AND π ARE INCOMMENSURABLE

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The purpose of this note is to give a short and elementary proof that, for p an odd integer greater than 1, $\theta = \arccos 1/p$ and π are incommensurable. The case $p=3$ is of importance in the theory of equivalence of polyhedra by dissection, and proofs for this special case, not so simple as the following, have been given [1, 2, 3]. On the other hand a more general theorem, embracing ours, but utilizing more advanced notions, has been given by D. H. Lehmer [4].

The proof proceeds by contradiction. Assume the existence of two relatively prime integers r and s such that

$$r\theta = s\pi. \tag{1}$$

Consider the binomial expansion

$$(\cos\theta + i\sin\theta)^r = \sum_{n=0}^{r}\binom{r}{n} i^n \cos^{r-n}\theta \sin^n\theta. \tag{2}$$

Taking the imaginary parts of both members, and remembering (1), we find

$$0 = \sin r\theta = \binom{r}{1}\cos^{r-1}\theta\sin\theta + \sum_{n=1}^{[(r-1)/2]}\binom{r}{2n+1}(-1)^n\cos^{r-(2n+1)}\theta\sin^{2n+1}\theta,$$

where $[(r-1)/2]$ denotes the largest integer not greater than $(r-1)/2$. Substituting $\cos\theta = 1/p$, $\sin\theta = p^{-1}(p^2-1)^{1/2}$, we find

$$0 = p^{-r}(p^2-1)^{1/2}\left\{r + \sum_{n=1}^{[(r-1)/2]}\binom{r}{2n+1}(-1)^n(p^2-1)^n\right\}.$$

Since p is an odd integer greater than 1, the quantity in braces must vanish, and r must be even inasmuch as each of the terms in the following sum is even. Since r and s were chosen as relatively prime, s must then be odd. Thus $\cos(r\theta/2) = \cos(s\pi/2) = 0$. Now, in equation (2), replace r by $r/2$ and take the real parts of both members, obtaining

$$0 = \cos(r\theta/2) = \cos^{r/2}\theta + \sum_{n=1}^{[r/4]} (-1)^n \binom{r/2}{2n} \cos^{r/2-2n}\theta \sin^{2n}\theta$$

$$= p^{-r/2}\left\{1 + \sum_{n=1}^{[r/4]} (-1)^n \binom{r/2}{2n} (p^2-1)^n\right\}.$$

But this equation is impossible because each term of the sum is even and the number in the braces is thus odd and cannot vanish. Therefore our original assumption (1) is untenable, and the theorem is proved.

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ON A PROBLEM OF ERDÖS

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In his neat solution of a problem proposed by Erdös, Lampek has shown [this MONTHLY, vol. 55, 1948, p. 103] that if $x_k = (k!)^2/\phi(k!)$, then $\phi(x_k) = k!$, ϕ being Euler's function. We have $x_1 = 1$, $x_2 = 2^2$, $x_3 = 2 \cdot 3^2$, $x_4 = 2^3 \cdot 3^2$, $\cdots$, and are led to notice that for each k the following result holds.

(1) *Either x_k has the property that each of its primes factors appears to the power two or higher, or $x_k = 2y_k$, where y_k is an odd integer having this property.*

Thus we are led to make the following definitions: P_i is the collection of all integers m such that exactly i distinct primes appear to the first power in the canonical factorization of m; $S_i(n)$ is the number of solutions $x \in P_i$ of the equation, $\phi(x) = n$.

Then (1) implies

(2) $$S_0(k!) \geqq 1.$$

I conjecture that also $S_1(k!) \geqq 1$, although I am not able to prove this. It may be of interest to note that for $2 \leqq k \leqq 6$, $S_1(k!)$ is the $(k-1)$th Fibonacci number; however, there is no obvious reason for expecting this to hold for larger values of k.

Investigating the functions S_i for more general arguments, we arrive at the following results:

(3) $$S_0(n) \leqq 1;$$

(4) $$\sup_n S_2(n) = \sup_n S_3(n) = \infty.$$

It follows from (2) and (3) that $S_0(k!)=1$.

It seems probable that $\sup_n S_j(n)=\infty$ for $j\geqq 1$.

Proof of (1). Observe that if k is prime, then $x_k=k^2x_{k-1}/(k-1)$, while $x_k=kx_{k-1}$ if k is composite. From this it follows by induction that if p is an odd prime and $p<k$, then $p^2 \mid x_k/k$, and (1) is an obvious consequence.

Proof of (3). Suppose that $y=\prod p_i^{a_i}\prod v_k^{c_k}$ and $z=\prod q_j^{b_j}\prod r_k^{d_k}$ are in P_0, the p's, v's, and q's denoting distinct primes. (The products involved may be empty, in which case they are defined to have the value 1.) Then if $\phi(y)=\phi(z)$ and $y\neq z$, we have

$$\prod p_i^{a_i-1} \mid \prod (q_j-1) < \prod q_j^{b_j-1} \mid \prod (p_i-1) < \prod p_i^{a_i-1},$$

where the inequalities are justified by the fact that all the exponents are $\geqq 2$. This clearly is a contradiction, so we must have $y=z$ and the proof is complete.

Proof of (4). This is an immediate consequence of results of Erdös, who has shown (Quart. Journ. Math., vol. 7, 1936, pp. 16 and 227) that the number of solutions of $\phi(x)=n$ which are products of two (three) distinct primes is unbounded as $n\to\infty$.

SOME EQUATIONS INVOLVING EULER'S TOTIENT FUNCTION

Leo Moser, University of Manitoba

We give some new results concerning certain equations involving $\phi(n)$ which have previously been discussed by V. L. Klee, Jr. [1].

The equation $\phi(n)=\phi(n+1)$ has the following solutions for $n<10{,}000$: $n=1$, 3, 15, 104, 164, 194, 255, 495, 584, 975, 2204, 2625, 2834, 3255, 3705, 5186, 5187. The last four entries are new and were found from Glaisher's table [2]. Klee noted that for $3<n<3000$ the odd numbers of n and $n+1$ are divisible by 15 but $n=5187$ shows that this is not always the case. Note that $n=5186$ is a solution of $\phi(n)=\phi(n+1)=\phi(n+2)$. P. Erdös mentioned the existence of such a solution in [3] and conjectured that for every k, $\phi(n)=\phi(n+1)=\phi(n+2)=\cdots=\phi(n+k)$ is solvable.

The equation $\phi(n)=\phi(n+2)$ is satisfied by $n=2(2p-1)$ if both p and $2p-1$ are odd primes, and by $n=2^{2^a+1}$ if $2^{2^a}+1$ is a Fermat prime. Other solutions with $n<10{,}000$ found from [2] are: $n=7$, 70, 308, 572, 635, 728, 910, 1015, 1330, 2132, 2170, 2590, 2695, 4292, 4338, 5950, 9100. Five of the entries under 3000 were overlooked in [1] while all those over 3000 are new.

The equation $\phi(n)+2=\phi(n+2)$ is satisfied if n and $n+2$ are primes, if n has the form $4p$ where p and $2p+1$ are primes, and if $n=2M_p$ where M_p is a Mersenne prime $M_p=2^p-1$. This last class of solutions was overlooked in [1]. We can show that there are no other solutions with $n<100{,}000$. To do this we make use of the following three lemmas:

1. *If* $\phi(n)+2=\phi(n+2)$ *then at least one of* n *and* $n+2$ *is of the form* p^a *or* $2p^a$ *where* p *is a prime of the form* $4r+3$.

Proof: If neither n nor $n+2$ is of this form then $\phi(n)$ and $\phi(n+2)$ would both be divisible by 4, so that their difference could not be 2.

2. *The case $\alpha=1$ leads to the classes of solutions mentioned above.*

Proof: If n is a prime then $\phi(n)=n-1$, while if n is composite then n has a prime factor $\leqq\sqrt{n}$, so that $\phi(n)\leqq n(1-1/\sqrt{n})=n-\sqrt{n}$. Hence if one of n and $n+2$ is prime so is the other. If $n=2p$, then $\phi(n)=p-1=(n-2)/2$, so that $\phi(n)+2=\phi(n+2)$ would imply $\phi(n+2)=(n+2)/2$, in which case $n+2$ is clearly a power of 2. If $n+2=2p'$ then $\phi(n+2)=p'-1=n/2$, so that $\phi(n)=n/2-2$, and n must be of the form $4p$.

3. *We have $\alpha\neq 2$.*

Proof: If $n=p^2$, then $\phi(n)=n-\sqrt{n}$, while if $n+2$ is composite (but clearly not a square), then $\phi(n+2)<(n+2)(1-1/\sqrt{n})^2<\phi(n)+2$. Similarly we can dispose of the cases $n=2p^2$, $n+2=p^2$ and $n+2=2p^2$.

This leaves relatively few numbers $<10^6$ to be examined and these can be tested directly.

References

1. V. L. Klee, this MONTHLY, vol. 54, 1947, p. 332.
2. J. W. L. Glaisher, Number Divisor Tables, Cambridge, 1940.
3. P. Erdös, Bull. Amer. Math. Soc. vol. 51, 1945, pp. 540–545.

CONGRUENCES FOR SETS OF PRIMES

P. A. CLEMENT, University of California, Los Angeles

1. Introduction. Wilson's function $P_1(n)$ is the function $P_1(n)\equiv(n-1)!+1$. By Wilson's theorem the condition $P_1(n)\equiv 0 \bmod n$ is necessary and sufficient in order that an integer $n>1$ be prime. In this note we find a congruence condition, similar to the above, for twin primality, and we indicate a method which furnishes a condition for sets of prime numbers of any prescribed type.

2. Twin primes. We shall establish the following result:

THEOREM. *A necessary and sufficient condition that two integers, n and $n+2$, $n>1$, both be prime is that*

$$4[(n-1)!+1]+n\equiv 0 \bmod n(n+2). \tag{1}$$

Proof. The sufficiency is obvious as divisions by n and $n+2$ separately reduce either to Wilson's theorem or to a simple modification of it.

The necessity follows as easily, but we wish to indicate how (1) may be obtain directly. Thus, with n and $n+2$ both primes, we have

$$(n-1)!+1\equiv 0 \bmod n, \tag{2}$$

$$(n+1)!+1\equiv 0 \bmod (n+2). \tag{3}$$

Reducing the factorial of (3) mod $(n+2)$ and rewriting as an equation we obtain

$$2[(n-1)!]+1=k(n+2), \qquad k \text{ some integer;} \tag{4}$$

then, using (2), we must have

$$2k+1\equiv 0 \bmod n. \tag{5}$$

Substitution of (5) in (4) determines the congruence of the theorem.

It may be noted that if 1 is considered the first prime, then the restriction $n>1$ can be deleted from the above theorem.

3. Further congruences. By analogous procedure, now using (1), we find that three positive integers n, $n+2$ and $n+6$, are a prime triple if and only if

$$4320[4(\overline{n-1!}+1)+n]+361n(n+2)\equiv 0 \bmod n(n+2)(n+6). \tag{6}$$

As stated, 1 is admitted as the first prime; if desired this may be obviated by requiring $n>1$. A similar congruence may be obtained for the other possible class of prime triples given by integers n, $n+4$, and $n+6$.

We indicate a less laborious method than that of the theorem for obtaining (6). By a modification of Wilson's theorem, $n+6$ is prime if and only if

$$720(n-1)!+1\equiv 0 \bmod (n+6). \tag{7}$$

Then using (1) we write

$$A[4(\overline{n-1!}+1)+n]+Bn(n+2)\equiv 0 \bmod n(n+2)(n+6),$$

and seek integers A and B so that this congruence mod $(n+6)$ reduces to a multiple of (7). This gives (5) immediately, and the process can be applied in this recursive fashion to prime sets of any prescribed type.

4. Prime quadruples. Let $P_2(n)$ be the function on the left of (1), and $P_3(n)$ be the left side of (6). We then have

$$P_2(n)=4P_1(n)+n,$$

and

$$P_3(n)=4320P_2(n)+361n(n+2).$$

The four positive integers n, $n+2$, $n+6$, $n+8$ may each be prime, the set then being a prime quadruple consisting of two sets of twin primes. For the function associated with this set, $P_4(n)$, we find

$$P_4(n)=224P_3(n)+111n(n+2)(n+6).$$

The congruence condition

$$P_4(n)\equiv 0 \bmod n(n+2)(n+6)(n+8)$$

is necessary and sufficient for the set to be a prime quadruple. By (1) a like condition is presented by the two congruences

$$P_2(n) \equiv 0 \bmod n(n+2)$$
$$P_2(n+5) \equiv 0 \bmod (n+6)(n+8).$$

As an exercise one might show that these two sets of conditions actually are equivalent.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

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LOGARITHMIC INTEGRATION*

H. F. MACNEISH, University of Miami

1. Introduction. Logarithmic differentiation is a device by means of which complicated products, quotients, and exponential functions may be differentiated with much less algebraic manipulation than is required by the use of the standard formula. We recall the rule for logarithmic differentiation:

$$\frac{dU(x)}{dx} = U(x)\frac{d}{dx}\ln U(x) \tag{1}$$

Applying this to a numerical example we have:

$$\frac{d}{dx}\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}} = \frac{x-x^2}{\sqrt{x^2+1}\,(x^3+1)^{4/3}}. \tag{2}$$

The integration of the answer, however, cannot be accomplished by standard methods. By integration we mean, as usual, the expression of the integral in a finite number of terms containing only elementary functions. In this note we outline a method for integrating certain expressions of this form, and we call the method "logarithmic integration."

2. Case I. Here we are concerned with a method of integrating certain ex-

* This paper was delivered before the New Haven meeting of the Association in September 1947 as a portion of the symposium "How to Solve it."

pressions of the type: $f(x) = [F_1(x)]^{p_1}[F_2(x)]^{p_2}R(x)$ where $F_1(x)$, $F_2(x)$, and $R(x)$ are polynomials and p_1 and p_2 are non-integral rational numbers. The extension to more factors $F_i(x)$ will be apparent. According to a theorem of Abel: If $\int f(x)dx$ is algebraic in x, then it is rational in $f(x)$ and x. (See J. F. Ritt "*Integration in Finite Terms*" p. 31.) This suggests that we consider as the simplest possibility:

$$\int f(x)dx = k[F_1]^{p_1+1}[F_2]^{p_2+1} + c \tag{3}$$

where k and c are constants.

Using logarithmic differentiation we have that:

$$f(x) = k[F_1]^{p_1}[F_2]^{p_2}\{(p_1 + 1)F_1'F_2 + (p_2 + 1)F_1F_2'\}.$$

Hence formula (3) holds if:

$$R(x) = k\{(p_1 + 1)F_1'F_2 + (p_2 + 1)F_1F_2'\}. \tag{4}$$

Consequently if (4) holds:

$$\int [F_1]^{p_1}[F_2]^{p_2}R\,dx = k[F_1]^{p_1+1}[F_2]^{p_2+1} + c. \tag{5}$$

Let us apply this to the example:

$$I = \int \frac{x^2 + 4x - 3}{(x + 1)^{1/2}(x^2 + 1)^{4/3}}\,dx$$

$$F_1 = x + 1; \qquad F_2 = x^2 + 1; \qquad R = x^2 + 4x - 3$$

$$p_1 = -1/2; \qquad p_2 = -4/3$$

$$(p_1 + 1)F_1'F_2 + (p_2 + 1)F_1F_2' = \frac{-x^2 - 4x + 3}{6}.$$

Hence (4) is satisfied with $k = -6$. Therefore

$$I = \frac{-6\sqrt{x + 1}}{\sqrt[3]{x^2 + 1}} + c.$$

If condition (4) fails to hold, the integration of $f(x)$ is not necessarily impossible, even as an algebraic function. For other rational combinations of x and $f(x)$ may be tried; for example consider expressions of the form $Q(x)[F_1]^{p_1+1}[F_2]^{p_2+1}$ where $Q(x)$ is a polynomial. The general question of the existence and determination of such an algebraic integral has been solved by Liouville (see Ritt, p. 32), but we shall not enter into this matter here. The question as to whether $\int f(x)dx$ is expressible in terms of elementary functions other than a'gebraic ones in an unsolved problem.

3. Case II. As a further illustration of this method let us consider the integral of $g(x)$ where $g(x)=[F(x)]^pR(x)$, $F(x)$ and $R(x)$ being polynomials and p a non-integral fraction. We assume here that the integral has the form:

$$\int g(x)dx = (ax+b)[F(x)]^{p+1} + c \tag{6}$$

where a, b, and c are constants. Differentiating we obtain:

$$[F(x)]^pR(x) = [F(x)]^p\{aF(x) + (p+1)(ax+b)F'(x)\}. \tag{7}$$

Hence formula (6) works if we can find a and b such that:

$$R(x) = aF(x) + (p+1)(ax+b)F'(x). \tag{8}$$

We illustrate with a numerical example:

$$I = \int \frac{5x^3 - 3x^2 + 6x - 3}{\sqrt{2x^3 + 4x - 1}}\,dx.$$

Then we must find a and b such that:

$$\begin{aligned} 5x^3 - 3x^2 + 6x - 3 &= a[2x^3 + 4x - 1] + \tfrac{1}{2}(ax+b)(6x^2+4) \\ &= 5ax^3 + 3bx^2 + 6ax + 2b - a. \end{aligned}$$

This is consistent in a and b, giving $a=1$ and $b=-1$. Hence:

$$I = (x-1)\sqrt{2x^3 + 4x - 1} + c.$$

SOME INTEGRAL FORMULAS

H. W. Smith, Oklahoma A. and M. College

When we orthonormalize the set of harmonic functions $r^n \cos n\theta$, $r^n \sin n\theta$ over the square of side 2 with center at the origin and sides parallel to the coordinate axes, integrals of the type

$$\int \sec^k\theta \cos p\theta d\theta, \qquad \int \sec^k\theta \sin p\theta d\theta$$

$$\int \csc^k\theta \cos p\theta d\theta, \qquad \int \csc^k\theta \sin p\theta d\theta$$

with appropriate limits must be computed.

Formulas 369–372 of Pierce's tables may be used as reduction formulas for these integrals, but they are given there as the sum of two other integrals of the same form.

To obtain reduction formulas that involve only one integral in the right members for the first two of these we integrate

$$\int \sec^k \theta e^{ip\theta} d\theta$$

by parts, to obtain

$$\int \sec^k \theta e^{ip\theta} d\theta = 1/ip \sec^k \theta e^{ip\theta} - k/ip \int \sec^k \theta \tan \theta e^{ip\theta} d\theta. \tag{1}$$

Integrating by parts again we find

$$\int \sec^k \theta \tan \theta e^{ip\theta} d\theta = 1/ip \sec^k \theta \tan \theta e^{ip\theta} - \frac{k+1}{ip} \int \sec^{k+2} \theta e^{ip\theta} d\theta + k/ip \int \sec^k \theta e^{ip\theta} d\theta.$$

Substitute this result into the right member of (1), solve for $\int \sec^{k+2} \theta e^{ip\theta} d\theta$ and separate the real and imaginary parts to obtain:

$$\int \sec^{k+2} \theta \cos p\theta d\theta = \frac{\sec^k \theta}{k(k+1)} [k \tan \theta \cos p\theta + p \sin p\theta] + \frac{k^2 - p^2}{k(k+1)} \int \sec^k \theta \cos p\theta d\theta.$$

$$\int \sec^{k+2} \theta \sin p\theta d\theta = \frac{\sec^k \theta}{k(k+1)} [k \tan \theta \sin p\theta - p \cos p\theta] + \frac{k^2 - p^2}{k(k+1)} \int \sec^k \theta \sin p\theta d\theta.$$

The last two integrals can be found in the same way and are:

$$\int \csc^{k+2} \theta \cos p\theta d\theta = \frac{-\csc^k \theta}{k(k+1)} [k \cot \theta \cos p\theta - p \sin p\theta] + \frac{k^2 - p^2}{k(k+1)} \int \csc^k \theta \cos p\theta d\theta.$$

$$\int \csc^{k+2} \theta \sin p\theta d\theta = \frac{-\csc^k \theta}{k(k+1)} [k \cot \theta \sin p\theta + p \cos p\theta] + \frac{k^2 - p^2}{k(k+1)} \int \csc^k \theta \sin p\theta d\theta.$$

The case appearing most frequently in the particular problem above is that in which $k = p = 2n$, so that in this case the coefficient of the integral on the right is zero.

REMARKS ON INTEGRATION BY PARTS

C. D. Olds, San Jose State College

This note originated in an attempt to show the calculus student that the formula for integration by parts is not merely a useful trick for finding anti-derivatives, but is somewhat deeper than that, and is, moreover, tied up intimately with the early history of elliptic integrals and functions, starting with the work of Fagnano (1682–1766) and leading up to the great discoveries of Abel. We also have here an example of a formula which leads to new results when in a certain situation it seems to be practically useless.

The formula for integration by parts shows that there is a mutual relation between the two integrals $\int ydx$ and $\int xdy$ by virtue of the equation

$$\int xdy + \int ydx = xy + c. \tag{1}$$

The advantage of this, of course, is that one integration may be substituted for the other. The student soon discovers that there are cases in which no alternation is produced by this substitution and that is when $x=f(y)$ and $y=f(x)$, that is, when x is the same function of y that y is of x. But even in this case the equation (1) is not entirely useless, for the equation

$$\int f(x)dx + \int f(y)dy = xy + c \tag{2}$$

still holds whether $\int f(x)dx$ can be integrated in finite terms or not, Clearly, the equations $x=f(y)$ and $y=f(x)$, manifestly imply that a symmetrical equation exists between x and y. Fagnano clearly perceived the importance of this simple remark and he deduced important consequences from it.

For example, let x be the abscissa of an hyperbola whose principal semi-axis is unity. Then its arc is given by

$$s = \int \left(\frac{e^2x^2 - 1}{x^2 - 1}\right)^{1/2} dx,$$

where e is the eccentricity. Let y be another abscissa, so related to the abscissa x that

$$ey = \left(\frac{e^2x^2 - 1}{x^2 - 1}\right)^{1/2},$$

or

$$e^2x^2y^2 = e^2(x^2 + y^2) - 1.$$

Since this equation is symmetrical with respect to x and y, it follows that

$$ex = \left(\frac{e^2y^2 - 1}{y^2 - 1}\right)^{1/2}$$

and hence from (2) that

$$\int \left(\frac{e^2x^2-1}{x^2-1}\right)^{1/2} dx + \int \left(\frac{e^2y^2-1}{y^2-1}\right)^{1/2} dy = exy + c.$$

The two integrals in this equation are elliptic and cannot be expressed in terms of a finite number of the elementary functions. Nevertheless the algebraic sum of two such integrals can be so expressed provided a symmetrical relation exists between x and y. Fagnano, about 1718 discovered many theorems of this type in his attempts to rectify the ellipse and the hyperbola. His methods are the starting point of the theory of elliptic functions, and are suggestive of those used by Euler in his proof (1761) of the addition theorem for elliptic integrals, this theorem in turn being a special case of Abel's famous theorem (1826) on the sum of Abelian integrals. Of course, how much of this can be made intelligible to the calculus student depends upon the knowledge and skill of the teacher. The writer has often been surprised at how much of this story can be told and he is convinced that opportunities like this one should not be overlooked. There is always the hope that by such discussions some of the teacher's enthusiasm for mathematics can be imparted to the student along with a little of the spirit for independent investigation. A few students, at least, will be encouraged to read more about the history of mathematics.

Equation (2) can be generalized so that three or more integrals are simultaneously combined. Thus, if two symmetrical equations exist between the variables x, y, z so that $yz=f(x)$, $xz=f(y)$, $xy=f(z)$, then

$$\int f(x)dx + \int f(y)dy + \int f(z)dz = xyz + c. \tag{3}$$

For example, if the two symmetrical equations are

$$x + y + z = a, \qquad x^2 + y^2 + z^2 = b^2,$$

then $f(x)=yz=1/2(a^2-b^2)-ax+x^2$, and (3) holds. It is an instructive exercise to verify this directly. Such a generalization is by no means as trivial as it seems, for it was in this manner that W. H. F. Talbot, the English pioneer of photography, was able to show that Abel's theorem is deducible from symmetric functions of the roots of equations and partial fractions. He was in possession of these results by 1825, about three years before Abel published his first theorem on the subject (1828).

References

1. W. H. F. Talbot, Researches in the Integral Calculus, Phil. Trans. Part 1, 1836, pp. 177–215.

2. One of Fagnanos' theorems is proved in Joseph Edwards, the Integral Calculus, vol. 1, Macmillan 1930, p. 582. See also G. N. Watson: The Marquis and the Land-Agent, Math. Gazette, vol. 17, 1933, pp. 5–17.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 844. *Proposed by Orrin Frink, Pennsylvania State College*

Sum the series

$$1 + 1/5! + 1/10! + 1/15! + \cdots + 1/(5n - 5)! + \cdots .$$

E 845. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

It is required to write the fifteen combinations of a, b, c, d in a sequence such that any two adjacent terms of the sequence shall differ by a single letter. How many such sequences are there? How can they be written down?

E 846. *Proposed by H. J. Hamilton, Pomona College*

The following is typical of many characterizations of the principal part of an infinitesimal which are to be found in elementary calculus texts.

"If an infinitesimal consists of two or more terms of different orders, the term of lowest order is called the *principal part* of the infinitesimal."

Show that this is not definitive and give a valid definition.

E 847. *Proposed by Albert Newhouse, University of Houston*

Let a, b, A be the given parts of a triangle in the ambiguous case. Show that the area of the triangle is given by

$$K = \tfrac{1}{2}b^2 \sin A[\cos A \pm (a^2 - b^2 \sin^2 A)^{1/2}].$$

E 848. *Proposed by Leo Moser, University of Manitoba*

Prove that every integer greater than 10^5 can be expressed as the sum of two abundant numbers.

E 849. *Proposed by C. W. Trigg, Los Angeles City College*

The area of a triangle is to the area of the triangle determined by the points of contact of its incircle (or excircle) as its circumdiameter is to its inradius (or exradius).

E 850. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Three persons, A, B, and C, in rotation throw a pair of dice. If the points 6, 7, 8 are consecutively gotten by A, B, C, then A is declared winner; if the points 7, 8, 6 are consecutively gotten by B, C, A, then B is declared winner; if

the points 8, 6, 7 are consecutively gotten by C, A, B, then C is declared winner. Find their chances of winning.

SOLUTIONS

Pythagorean Triangles with Equal Perimeters

E 812 [1948, 248]. *Proposed by Monte Dernham, San Francisco, California*

Find the shortest perimeter common to two different primitive Pythagorean triangles.

Solution by Fritz Herzog, Michigan State College. To each primitive Pythagorean triangle corresponds bi-uniquely a pair of integers s, t with $s>t>0$, $(s, t)=1$, and $s\not\equiv t \pmod 2$, such that the sides of the triangle are $2st$, s^2-t^2, s^2+t^2. (See Hardy and Wright, *An Introduction to the Theory of Numbers*, Theorem 225, p. 189.) The perimeter of the triangle is $2su$, where $u=s+t$, and the above conditions on s and t read, in terms of s and u:

$$(1) \qquad s < u < 2s; \qquad (s, u) = 1; \qquad u \equiv 1 \pmod 2.$$

The problem is, therefore, to find the smallest positive integer y which admits of two different representations $y=su$, where s and u satisfy (1). Let $y=su$ and $y=s'u'$ be two such representations and assume that $u<u'$ and hence $s>s'$. Let c/d be the reduced form of the fraction $u/u'=s'/s$ and write $u=bc$, $u'=bd$, $s=ad$, $s'=ac$. The conditions (1) on s, u and s', u' together with $u<u'$ are easily shown to be equivalent to the following conditions on the four positive integers a, b, c, d:

$$(2) \qquad d/2c < a/b < c/d < 1,$$

(3) a, b, c, d mutually relatively prime,

$$(4) \qquad b \equiv c \equiv d \equiv 1 \pmod 2.$$

Since $y=abcd$ it remains to solve the inequality (2) with the restrictions (3) and (4) and such that $y=abcd$ is a minimum.

We first conclude from (2) that $1/2<a/b<1$. Hence the values of a and b which give the smallest possible value of ab are $a=2$, $b=3$. Let a and b be so chosen. Then from (2), $3/4<c/d<1$ and, in view of (3) and (4), the values of c and d which give the smallest possible value of cd (after the choice $a=2$, $b=3$) are $c=11$, $d=13$. We thus arrive at the solution $a=2$, $b=3$, $c=11$, $d=13$, which yields $y=858$.

To show that there is no solution of (2), (3), and (4) which would result in a smaller value of y, we observe first that, by (2), $1/2<c^2/d^2<1$. The only values of c and d satisfying this inequality as well as (3) and (4) and yielding a value cd smaller than $11\cdot 13$ are (i) $c=9$, $d=11$, (ii) $c=7$, $d=9$, and (iii) $c=5$, $d=7$. However, as was shown above, in these three cases b could not equal 3 so that, by (3) and (4) $b\geqq 5$ in (i) and (ii) and $b\geqq 9$ in (iii). Thus by (2) we obtain for all

three of the above choices of c and d

$$y = abcd > (bd/2c)bcd = (bd)^2/2 \geqq (9\cdot 5)^2/2 > 858.$$

Thus the smallest number having the property required in the proposed problem is $2\cdot 858 = 1716$, which is the perimeter of the primitive Pythagorean triangles 364, 627, 725 and 748, 195, 773.

Also solved by W. E. Buker, William Douglas, L. S. Kennison, Sidney Kravitz, Roger Lessard, D. W. Matlack, F. L. Miksa, and the proposer.

Several solvers resorted to more or less laboriously constructed tables. Miksa found thirty-six perimeters, less than 20000, which are common to two different primitive Pythagorean triangles. He also found the perimeter 14280 as the only one, less than 20000, which is common to three primitive Pythagorean triangles. C. S. Ogilvy found 240 as the smallest perimeter common to two non-primitive Pythagorean triangles. This result was also given by Buker who, in addition, found 1680 as the smallest perimeter common to three non-primitive Pythagorean triangles.

Editorial Note. For other problems involving Pythagorean triangles see E 18, E 67, E 73, E 283, E 324, E 327, E 380, and E 828.

Magic Square as a Determinant

E 813 [1948, 248]. *Proposed by C. W. Trigg, Los Angeles City College*

Let S be the sum of the integer elements of a magic square of order three, and let D be the value of the square considered as a determinant. Show that D/S is an integer.

I. *Solution by R. J. Walker, Cornell University.* Let

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

have the magic sum $N = S/3$. Then

$$N = (a+e+i) + (d+e+f) + (g+e+c) - (a+d+g) - (c+f+i) = 3e,$$

and $S = 9e$. Hence, adding rows and columns,

$$D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ 3e & 3e & 3e \end{vmatrix} = \begin{vmatrix} a & b & 3e \\ d & e & 3e \\ 3e & 3e & 9e \end{vmatrix} = \begin{vmatrix} a & b & e \\ d & e & e \\ 1 & 1 & 1 \end{vmatrix} S.$$

II. *Solution by Monte Dernham, San Francisco.* Every magic square of the third order may be written in a form equivalent to

$$\begin{array}{lll} m+x & m-x-y & m+y \\ m-x+y & m & m+x-y \\ m-y & m+x+y & m-x \end{array}$$

(See Kraitchik, *Mathematical Recreations*, p. 148.) Considering the square as a determinant, expanding and simplifying, we find its value to be

$$9m(y^2 - x^2);$$

and, since $S=9m$, and all elements are integers, D/S is an integer.

If the square be transformed by rotation, the foregoing result remains unaffected; likewise, if it be transformed by reflection in a mirror, except that in the latter case x^2 and y^2 are interchanged.

III. *Remarks by the Proposer.* The property $D/S=I$, an integer, may be extended to magic squares of higher orders with certain restrictions.

In the absence of a qualifying adjective, such as pandiagonal, semi-nasik, multiplicative, *etc.*, there is a certain looseness in the use of the term "magic square." Sometimes the term is defined as a square array of integers with the property that the elements of each row, of each column, and of each of the principal diagonals have equals sums (*cf.* E 791). As shown by the preceding proofs, $D/S=I$ for all third order squares of this broad type.

An nth order square, for which $D/S=I$, may have this property spoiled without loss of its magical nature by adding the proper integer to each of n properly chosen elements. Let the coordinates of an element be determined by its row and column. Then, if n is even, the square may be spoiled by adding an integer x to the n elements (1, 1), (2, n), (3, 2) (4, 2), $\cdots$, (n, $n-1$). If $n>3$ is odd, the square may be spoiled by adding x to the n elements (1, 1), (2, n), (3, $n-1$), (4, 2), (5, 3), $\cdots$, (n, $n-2$). For example, consider the squares:

$$\begin{array}{cccc} \underline{15} & 6 & 9 & 4 \\ 2 & 13 & 8 & \underline{11} \\ 7 & \underline{12} & 1 & 14 \\ 10 & 3 & \underline{16} & 5 \end{array} \qquad \begin{array}{ccccc} \underline{17} & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & \underline{16} \\ 4 & 6 & 13 & \underline{20} & 22 \\ 10 & \underline{12} & 19 & 21 & 3 \\ 11 & 18 & \underline{25} & 2 & 9 \end{array}$$

The fourth order square has $D/S=-96$. When x is added to each of the underlined elements the derived square is still magic, but now

$$D/S = (x + 16)(x^2 + 4x - 24)/4,$$

which is an integer only if x is even. The fifth order square has $D/S=15600$. When x is added to each of the underlined elements we find

$$D/S = (x^4 + 25x^3 + 60x^2 - 3475x - 78000)/5,$$

which is an integer only when x is a multiple of 5. It may be conjectured that if x is prime to n, D/S, is not an integer.

More frequently, an nth order "magic square" is defined as above with the restriction that its elements are the first n^2 integers. Occasionally this restriction is weakened to include any n^2 integers in arithmetic progression. Under this latter condition D/S may be examined more profitably. Let the difference in the arithmetic progression be d, and let Y be the sum of the elements of any row or column. Then

$$S = nY = (n^2/2)[2a + (n^2 - 1)d],$$

whence

$$Y = (n/2)[2a + (n^2 - 1)d] = n[a + (d/2)(n^2 - 1)].$$

If n is odd, n^2-1 is even, and $Y=nk$, where k is an integer. If n is even and d is even, then again $Y=nk$. In the determinant add the elements of the first $n-1$ columns to those of the nth column. In the derived determinant add the elements of the first $n-1$ rows to those of the nth row. Every element of the nth row and nth column is now $Y(=nk)$ except the common element of this row and column, which is nY. Hence Y and n may be factored out, so that $D=nYD'$ and $D/S=D'$, which is an integer since the elements of the determinant D' are integers. It will be observed that no use has been made of the sum of the diagonals in the proof.

The case where n is even and d is odd offers more difficulty, for then $Y=nk/2$, where k is odd, so that

$$D = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} \cdot & \cdot & \cdot & Y \\ \cdot & \cdot & \cdot & Y \\ \cdot & \cdot & \cdot & Y \\ Y & Y & Y & nY \end{vmatrix} = (Yn/2) \begin{vmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 \\ k & k & k & 2 \end{vmatrix} = YnD'/2.$$

Then, in order that D/S may be an integer, D' must be even.

In the evaluation of these determinants the rows may be interchanged at random, as may be the columns, since only the absolute value is of concern. Since n is even and d is odd, there will be $n^2/2$ even integers (e) and $n^2/2$ odd integers (o) in the array. If in any pair of columns (or rows) the odd and even integers occurs in the same sequence we shall say that the pair is "matched," *e.g.*,

$$\begin{matrix} o & o & e & o & e & e & o & e \\ o & o & e & o & e & e & o & e \end{matrix}.$$

If in any pair of columns (or rows) the odd and even integers occur completely

out of sequence we shall say that the pair is "mismatched," *e.g.*,

$$\begin{matrix} o & o & e & o & e & e & o & e \\ e & e & o & e & o & o & e & o \end{matrix}.$$

If in a pair of columns (or rows) reference is made to two integers in the same row (or column) they will be called a "couple."

If the pair under consideration are rows, interchange the columns and rows. If a pair of columns are matched or matched except for a single couple, rearrange the columns so that neither of the pair is the nth column. Rearrange the rows so that the exceptional couple falls in the nth row. Convert D into $YnD'/2$. In D' add the elements of one of the pair to the elements of the other of the pair, which will then contain all even elements, so D' is even. (All even-ordered magic squares which I have examined fall in this category.)

If the pair is mismatched or mismatched except for a single couple, prepare the determinant as in the case of the matched pair. Convert D into $YnD'/2$. In D' successively add each column of the pair to the nth column, which will then contain all even elements, so D' is even.

It may be noted further that if n is of the form $4m$, then Y is even, so in each of the columns, rows, and diagonals the odd and even elements are each even in number. If n is of the form $4m+2$, then Y is odd, so each column, row, and diagonal contains an odd number of odd elements.

When $n=4$, all elements of a particular row or column will be like or there will be two odd and two even elements. There are 6 permutations of *eeoo*, and 15 combinations of distinct permutations taken 4 at a time. (No combination containing two like permutations need be considered since they constitute a matched pair for which D' is even.) Only three of these combinations contain an even number of odd and an even number of even elements in every column. Hence, by interchanging rows, all possible squares devoid of matched pairs may be converted into one of the following:

$$\begin{matrix} e & e & e & e \\ o & o & o & o \\ e & e & o & o \\ o & o & e & e \end{matrix} \qquad \begin{matrix} e & e & o & o \\ e & o & e & o \\ o & e & o & e \\ o & o & e & e \end{matrix} \rightarrow \begin{matrix} e & e & o & o \\ e & o & o & e \\ o & e & e & o \\ o & o & e & e \end{matrix} \rightarrow \begin{matrix} e & o & e & o \\ e & o & o & e \\ o & e & e & o \\ o & e & o & e \end{matrix}$$

Each of these squares contains a mismatched pair, so D' is even and $D/S=I$ for all fourth order squares composed of 16 elements in arithmetic progression. (This fact has been established in another way by this writer in a note on *Determinants of fourth order magic squares*, this MONTHLY, November, 1948.) Since the sum of the elements of the diagonals has not been employed, the squares need be magical only in their columns and rows.

When $n=6$, an arrangement is possible which may be magical in rows, columns, and diagonals, although I have no numerical example of this, namely:

$$\begin{array}{cccccc} o & e & e & o & o & e \\ o & o & e & e & o & e \\ e & o & o & e & o & e \\ o & e & o & e & e & o \\ e & o & e & e & o & o \\ e & e & o & e & o & o \end{array}$$

When this is converted into $YnD'/2$ and D' is expanded by minors, D' is found to be odd. Hence this approach will not confirm the integer nature of D/S for magic squares with $n>4$ and even and d odd without calling upon some of the other properties of these magic squares.

Also solved by R. V. Andree, W. G. Brady, J. C. Eaves, R. W. Frankel, R. E. Harton, Roger Lessard, N. S. Mendelsohn, Leo Moser, S. T. Parker, and T. L. Reynolds. Several solvers gave partial generalizations.

Location of a Minimum Area

E 814 [1948, 248]. *Proposed by Sidney Kravitz, New York, New York*

Given the curve $y=e^x/x$. Consider all areas under the curve, over the x-axis, and between two ordinates one unit apart. Locate the boundary lines of the area which is a minimum.

Solution by D. W. Matlack, North American Aviation, Inc. Let the area under consideration be A. Then

$$A = \int_x^{x+1} y\,dx,$$

and

$$dA/dx = y\Big|_x^{x+1} = \frac{e^{x+1}}{x+1} - \frac{e^x}{x}.$$

Setting $dA/dx=0$ we find $x=1/(e-1)$, $x+1=e/(e-1)$.

Also solved by W. G. Brady, Karl Itkin, J. N. P. Lawrence, Roger Lessard, Julius Liebleim, Leo Moser, C. S. Oglivy, S. T. Parker, and the proposer.

Ogilvy stated the general theorem: *Of all areas bounded by the curve $y=f(x)$, the x-axis, and two ordinates n units apart, the maximum (or minimum) is that whose left-hand ordinate is the solution of x of the equation $f(x+n)-f(x)=0$.* It is interesting to note that no matter how unsymmetrical or irregular the curve may be between x and $x+n$, the two bounding ordinates of the required area are of equal length.

J. N. Eastham pointed out that this problem occurs as Ex. 12, p. 249, in Reddick and Miller, *Advanced Mathematics for Engineers*, 2nd ed.

Several solvers showed that the minimum of the given curve occurs at $x=1$.

The area in question is approximately equal to 2.83 square units.

Marble in Bowl

E 815 [1948, 248]. *Proposed by J. S. Miller, Dillard University*

A marble rolls in a hemispherical bowl. Find its period.

Solution by J. C. Miller, University of California, Berkeley. Denote the radius of the bowl by R, the radius and mass of the marble by r and m respectively, and the acceleration of gravity by g. Then in spherical coordinates, with the pole directed downwards, the kinetic and potential energies T and V are

$$T = (7/10)m(R - r)(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)$$

and

$$V = mg(R - r)(1 - \cos \theta).$$

The Lagrangian equations of motion are thus

$$\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta + [5g/7(R - r)] \sin \theta = 0 \tag{1}$$

and

$$d(\dot{\phi} \sin^2 \theta)/dt = 0. \tag{2}$$

Equation (2) integrates immediately to give

$$\dot{\phi} \sin^2 \theta = \text{constant} = a.$$

Substitution of this solution in equation (1) gives a first integral

$$\dot{\theta}^2 + a^2/\sin^2 \theta - [10g/7(R - r)] \cos \theta = \text{constant} = b. \tag{3}$$

To avoid elliptic integrals keep θ small and replace $\sin \theta$ by θ, and $\cos \theta$ by $1-\theta^2/2$. Equation (3) may then be integrated for time to give

$$t = \int_0^{\theta} \theta(-a^2 + c\theta^2 - \omega^2\theta^4)^{-1/2} d\theta,$$

where

$$c = a^2/\theta_0^2 + \omega^2\theta_0^2 \quad \text{and} \quad \omega^2 = 5g/7(R - r),$$

provided the motion starts from rest at $\theta = \theta_0$. The period Ω will be the time required for θ to go from θ_0 to $-\theta_0$ and back, so that

$$\begin{aligned} \Omega &= 4 \int_0^{\theta_0} \theta(-a^2 + c\theta^2 - \omega^2\theta^4)^{-1/2} d\theta \\ &= (2/\omega) \arcsin \left[(2\omega^2\theta^2 - c)(c^2 - 4a^2\omega^2)^{-1/2}\right]\Big|_0^{\theta_0} \\ &= \pi/\omega - (2/\omega) \arcsin \left[c(c^2 - 4a^2\omega^2)^{-1/2}\right]. \end{aligned}$$

For $\dot{\phi}=0$, *i.e.* $a=0$, the period will be

$$2\pi/\omega - 2\pi[7(R-r)/5g]^{1/2},$$

which is the time for one complete oscillation in a plane through the polar axis. For large θ the time integral will be an elliptic integral of the first kind.

Also solved by Roger Lessard, who pointed out that this problem may be found in Timoshenko and Young, *Engineering Mechanics*, problem 534, page 463. It is there shown that the system is equivalent to a simple pendulum whose length is $7(R-r)/5$.

Editorial Note: This problem is not to be confused with that where a particle *slides* from the edge to the bottom of a smooth hemispherical bowl of radius R. This latter problem furnishes a nice application of the gamma function. The time of descent of the particle is

$$t = [\Gamma(1/4)/\Gamma(3/4)][\pi R/8g]^{1/2}.$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4325. *Proposed by Orrin Frink, Pennsylvania State College*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

4326. *Proposed by R. J. Walker, Cornell University*

Prove or disprove: 128 is the only power of 2 of two or more digits each of which is a power of 2.

4327. *Proposed by R. J. Walker, Cornell University*

In attempting to solve the preceding problem we tried to show that for some positive integer r no power of 2 had its last r digits all powers of 2. Show that this attempt had to fail; in particular, show that for each r there exist powers of 2 each of whose last r digits is either 1 or 2.

4328. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a triangle ABC whose altitudes are AA', BB', CC'. Prove that the Euler lines of the triangles $AB'C'$, $BC'A'$, $CA'B'$ are concurrent on the nine-point circle at a point P which is such that one of the distances PA', PB', PC' equals the sum of the other two.

4329. *Proposed by A. W. Goodman, Rutgers University*

Let θ be an irrational number, $a=e^{i\theta\pi}$. Prove that

$$f(z) = \sum_{n=0}^{\infty} a^{n^2} z^n$$

has the unit circle as a natural boundary.

4330. *Proposed by Paul Erdös, Syracuse University*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers. Prove that there exists either an infinite subsequence in which no integer divides another or an infinite subsequence where each integer is a multiple of the preceding one.

SOLUTIONS

A Permutation Problem

3731 [1935, 255]. *Proposed by Raphael Robinson, University of California at Berkeley*

In how many ways can a_1 1's, a_2 2's, $\cdots$, a_n n's be arranged, so that in reading from the beginning, none of the $(k+1)$'s are reached until at least one of the k's has been reached?

Solution by W. D. Smith, Student, Rutgers University. All desired arrangements may be obtained in the following way: (A) place a 1 first and put the other 1's in any selection of a_1-1 out of the $a_1+a_2+\cdots+a_n-1$ remaining places; then (B) place a 2 in the first remaining vacant place and put the other 2's in any selection out of the $a_2+a_3+\cdots+a_n-1$ remaining places; then (C) continue in the same fashion with the 3's, 4's, $\cdots$ until the $(n-1)$'s have been placed; and finally (D), the a_n n's will fit into the remaining a_n places in just one way. The number of ways in which each step can be taken is clearly:

$$\text{(A)} \quad \binom{a_1 + a_2 + \cdots + a_n - 1}{a_1 - 1}, \quad \text{(B)} \quad \binom{a_2 + a_3 + \cdots + a_n - 1}{a_2 - 1},$$

$$\text{(C)} \quad \binom{a_3 + \cdots + a_n - 1}{a_3 - 1}, \cdots, \binom{a_{n-1} + a_n - 1}{a_{n-1} - 1},$$

where $\binom{x}{y}$ is the number of combinations of x things taken y at a time. Their product is the total number of ways of forming the desired arrangement:

$$\text{(1)} \qquad \prod_{k=1}^{n} \binom{a_k + a_{k+1} + \cdots + a_n - 1}{a_k - 1}.$$

This result may also be put in the following form:

$$\frac{(a_1 + a_2 + \cdots + a_n)!}{\prod_{k=1}^{n} (a_k + a_{k+1} + \cdots + a_n) \cdot (a_k - 1)!}.$$

If instead of a_k k's ($k = 1, 2, \cdots, n$) we had been given a_k members of a kth class distinguishable one from another, the total number of ways would be given by (1) multiplied by $a_1!a_2! \cdots a_n!$.

A Locus of Radical Centers

4200 [1946, 225]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given in a plane the triangle ABC and the fixed point P which is center of a variable circle (P). Find the locus of the radical center of the circles passing through A, B, C, respectively, which have with (P) the sides BC, CA, AB as radical axes. Consider the analogous problem for a tetrahedron and a sphere with fixed center, and show that the locus is a twisted cubic through the vertices and centroid of the tetrahedron.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** Let ABC be the triangle of reference in a system of trilinear coördinates. A circle (P) is given by

$$(P) \equiv (ayz + bxz + cxy) + K(ax + by + cz)(ux + vy + wz) + \lambda(ax + by + cz)^2 = 0.$$

A circle (A), having with (P) the radical axis AB, is given by

$$(P) + \mu x(ax + by + cz) = 0.$$

It goes through A if $Kau + \lambda a^2 + \mu a = 0$, whence $\mu = -(Ku + \lambda a)$ and

$$(A) \equiv (P) - (Ku + \lambda a)(x)(ax + by + cz).$$

The radical center of (A) and the analogous circles (B) and (C) is given by

$$(Ku + \lambda a)x = (Kv + \lambda b)y = (Kw + \lambda c)z.$$

It describes a conic whose equation is

$$\begin{vmatrix} u & a & 1/x \\ v & b & 1/y \\ w & c & 1/z \end{vmatrix} = 0$$

or

$$yz(cv - bw) + xz(aw - cu) + xy(bu - av) = 0.$$

This conic is easily seen to be circumscribed about the triangle ABC and to pass

* Translated from the French by O. J. Ramler, Catholic University of America, Washington, D. C.

through its center of gravity. It could be defined as follows:

If O is the center of the circle ABC, let U be the point in which a perpendicular to OP meets the ideal line; let V be the isogonal conjugate of U, then V is on the circle ABC; then the above conic is the triangular inverse of the trilinear polar of V.

In the case of a tetrahedron $ABCD$, let $(S)=0$ be the equation of the circumsphere and let $(\pi)\equiv Ax+By+Cz+Dt$. Then a sphere (P) is given by

$$(P) \equiv (S) + K(\pi)(ux + vy + wz + st) + \lambda(\pi)^2 = 0.$$

An arbitrary sphere, having with (P) the radical plane BCD is given by $(P)+\mu x(\pi)=0$. It goes through A if $\mu=-(Ku+\lambda A)$, where A is the area of BCD, B is the area of ACD, $\cdots$. The radical center of the four analogous spheres is given by

$$x(Ku + \lambda A) = y(Kv + \lambda B) = z(Kw + \lambda C) = t(Ks + \lambda D).$$

Its locus is the twisted cubic which is circumscribed about the given tetrahedron and which is the intersection of the cones

$$\begin{vmatrix} u & A & 1/x \\ v & B & 1/y \\ w & C & 1/z \end{vmatrix} = 0, \qquad \begin{vmatrix} u & A & 1/x \\ v & B & 1/y \\ s & D & 1/t \end{vmatrix} = 0.$$

This cubic goes through the center of gravity of the tetrahedron.

Divisibility of Factorial Numbers

4252. [1947, 287]. *Proposed by Paul Erdös, Syracuse University*

It is well known that $2n!/n!(n+1)!$ is always an integer. Prove that for every k there are infinitely many n's such that $2n!/n!(n+k)!$ is an integer.

Solution by Fritz Herzog, Michigan State College. Let k be a fixed integer greater than unity. For integral m and integral $q>1$, we put

$$F(m, q) = [2m/q] - [m/q] - [(m + k)/q], \tag{1}$$

where $[x]$ represents the greatest integer which does not exceed x. It then follows from a well known formula in number theory that $(2n)!/n!(n+k)!$, for integral $n\geqq 0$, is an integer if and only if

$$\sum_{a=1}^{\infty} F(n, p^a) \geqq 0 \tag{2}$$

for all primes p.

We shall need the following facts concerning $F(m, q)$. In the first place, we have obviously $F(m, q)=F(m', q)$ when $m\equiv m' \pmod q$. Secondly, we show that

(3) $F(m, q) \geqq 0$ for $q \geqq 2k$ and for all m.

By the above, it suffices to assume that $0 \leqq m < q$, so that $[m/q]=0$. For $0 \leqq m < q-k$ we have $[2m/q]=0$ or 1 and $[(m+k)/q]=0$; for $q-k \leqq m < q$ we have $[2m/q]=1$ and $([m+k)/q]=1$. Thus (3) follows from (1). Thirdly we show that

(4) $F(m, q) = 1$ for $q \geqq 2k+2$ and for $m \equiv -k-1 \pmod{q}$.

For we have $F(m, q)=F(-k-1, q)$ and it is easily verified that $[(-2k-2)/q] = [(-k-1)/q]=[(-1)/q]=-1$. Thus (4) follows from (1).

From (3) we conclude that (2) holds for all n when p is a prime greater than $2k$. Let p be a prime less than $2k$, let A be such that $p^{A+1} \geqq 2k$ and let $n \equiv -k-1 \pmod{p^A}$. Then, by (3), $F(n, p^a) \geqq 0$ when $a > A$ and, by (4), $F(n, p^a)=1$ when $2k+2 \leqq p^a \leqq p^A$. Thus we may choose A sufficiently large, with $n \equiv -k-1 \pmod{p^A}$, so that there will be enough values of a for which $F(n, p^a)=1$ to cancel out the effects of those values of a less than $2k+2$ for which perhaps $F(n, p^a)<0$, whence (2) will hold for p, the particular prime under consideration. We now apply this process to each prime $p<2k$, obtaining in each case an exponent $A=A(p)$. Let P be the product of all prime powers $p^{A(p)}$ thus obtained. Then (2) holds for $n \equiv -k-1 \pmod{P}$ and for all primes p. Hence $(2n)!/n!(n+k)!$ is an integer for at least every $n \equiv -k-1 \pmod{P}$.

Also solved by P. T. Bateman, Robert Breusch, R. H. Bruck, and N. J. Fine.

Editorial Note. The Proposer added the following remarks. (a) It can be shown that the values of n for which $(2n)!/n!(n+k)!$ is not an integer have density zero. More precisely, the number of $n \leqq x$ for which $(2n)!/n!(n+k)!$ is not an integer is less than $x^{1-\epsilon}$ for large x. (b) Balakran proved (Journal of the Indian Mathematical Society, vol. 1) that for infinitely many n, $(2n)!/(n+1)! \cdot (n+1)!$ is an integer. A more difficult problem is to decide whether there are infinitely many n for which $(2n)!/(n+2)!(n+2)!$ is an integer.

Limit of a Sequence

4255 [1947, 346]. *Proposed by G. Polya, Stanford University.*

A sequence $[x_n]$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{(n-1)g}{1+(n-1)g} x_{n-1} + \frac{1}{1+(n-1)g} x_{n-2},$$

where g is a given positive quantity. Find an expression for the limit of x_n as $n \to \infty$. (This generalizes problem E 694 (1945, 516) which corresponds to the special case $g=1$.)

Solution by R. C. Buck, Brown University. We solve a somewhat more general problem first. Let $p_n > 1$ be a given sequence of positive numbers such that

$$\lim_{n\to\infty} \prod_{k=2}^{n} p_k = +\infty.$$

Define the sequence $\{x_n\}$ by

$$x_n = x_{n-1}(p_n - 1)/p_n + x_{n-2}/p_n.$$

Then $\{x_n\}$ converges to the number $x_1-(x_1-x_0)K$, where K is a constant depending only on $\{p_n\}$.

To see this, we rewrite the recursion formula as

$$x_n - x_{n-1} = (x_{n-1} - x_{n-2})/(-p_n);$$

by iteration, this becomes

$$x_n - x_{n-1} = (-1)^{n-1}(x_1 - x_0)/p_2p_3 \cdots p_n,$$

and summing,

$$x_n = x_1 + (x_1 - x_0)\sum_{k=2}^{n} (-1)^{k-1}/p_2p_3 \cdots p_k.$$

By virtue of the assumptions on p_n, the series

$$\sum_{n=2}^{\infty} (-1)^n/p_2p_3 \cdots p_n$$

is a convergent alternating series. If we denote its sum by K, then the sequence $\{x_n\}$ is convergent to the limit $x_1-(x_1-x_0)K$.

Other expressions for K are possible. Thus, by a familiar transformation, K can be expressed as the continued fraction

$$K = \frac{1}{p_2 + \dfrac{p_2}{p_3 - 1 + \dfrac{p_3}{p_4 - 1 + \dfrac{p_4}{p_5 - 1 + \cdots}}}}.$$

In the special case of the proposed problem, $p_n=1+(n-1)g$ and K can also be expressed as a definite integral. Let

$$F(x) = \frac{x}{1+g} - \frac{x^2}{(1+g)(1+2g)} + \frac{x^3}{(1+g)(1+2g)(1+3g)} - \cdots.$$

Then, $(1+x)F(x)+gx\,F'(x)=x$, and solving this we have

$$F(x) = c(xe^x)^{-c}\int_0^x (ue^u)^c du,$$

where $c=1/g$. Hence

$$K = F(1) = ce^{-c}\int_0^1 (ue^u)^c du.$$

Also solved by P. T. Bateman, N. J. Fine, J. G. Herriot, Aaron Herschfeld, J. Lehner and A. M. Peiser, J. F. Locke, Leo Moser, and the Proposer.

Editorial Note. The Proposer's solution is in the form

$$\lim_{n\to\infty} x_n = x_0 + (x_1 - x_0)\int_0^1 \exp\left(\frac{t^g - 1}{g}\right)dt.$$

Circles Orthogonal to a Sphere

4256 [1947, 346]. *Proposed by N. A. Court, University of Oklahoma*

Given a sphere orthogonal to two circles lying in two distinct planes. If the center of the sphere is conjugate, with respect to one of the circles, to the point in which the plane ot that circle cuts the axis of the other circle, the same is true of the center of the sphere, if the roles of the two circles are interchanged.

Note. A circle is orthogonal to a sphere if the plane of the circle cuts the sphere along a great circle orthogonal to the given circle (see, for instance, the Proposer's *Modern Pure Solid Geometry*, p. 138, art. 416).

Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C. Let the sphere have center O, radius R: and the circles have centers O_1, O_2 and radii r_1, r_2, respectively. The axis of (O_1) meets the plane of (O_2) in the point P_2. The corresponding point in the plane of (O_1) is P_1. From the given orthogonal property,

$$\overline{OO_1}^2 = r_1^2 + R^2, \qquad \overline{OO_2}^2 = r_2^2 + R^2. \tag{1}$$

Since the axes are perpendicular to the planes of their circles,

$$\overline{P_1P_2}^2 = \overline{P_1O_1}^2 + \overline{P_2O_1}^2 = \overline{P_1O_2}^2 + \overline{P_2O_2}^2, \tag{2}$$

$$\overline{P_1O}^2 = \overline{OO_2}^2 + \overline{P_1O_2}^2, \qquad \overline{P_2O}^2 = \overline{OO_1}^2 + \overline{P_2O_1}^2. \tag{3}$$

Consider first the point P_1. The powers of P_1 with respect to the great circle section of the sphere (O) and with respect to its orthogonal circle (O_1) are respectively

$$p_1 = \overline{P_1O}^2 - R^2, \qquad p_2 = \overline{P_1O_1}^2 - r_1^2. \tag{4}$$

From the first of equations (1), equation (2), and the second of equations (3) find

$$p_2 = \overline{P_1O_1}^2 - r_1^2 = \overline{P_1O_2}^2 + \overline{P_2O_2}^2 - \overline{P_2O}^2 + R^2. \tag{5}$$

From the second of equations (1) and the first of equations (3) find

$$p_1 = \overline{P_1O}^2 - R^2 = r_2^2 + \overline{P_1O_2}^2. \tag{6}$$

The condition that P_1 be the conjugate of O with respect to (O_1) is that the powers of (4) should be equal. With $p_1 = p_2$, (5) and (6) become

$$\overline{P_2O}^2 - R^2 = \overline{P_2O_2}^2 - r_2^2,$$

which is exactly the condition that P_2 should be the conjugate of O with respect to (O_2), which proves the theorem.

Also solved by Ou Li, and the Proposer.

Triangle and Orthocenter

4258 [1947, 346]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the necessary and sufficient condition that four non-collinear points are such that each is the orthocenter of the other three, is

$$\pm 34 \cdot 42 \cdot 23 \pm 41 \cdot 13 \cdot 34 \pm 12 \cdot 24 \cdot 41 \pm 23 \cdot 31 \cdot 12 = 0,$$

where rs denotes the distance between the rth and sth points, and three of the signs differ from the fourth.

Solution by the Proposer. Let the four points be designated by a, b, c, d in a system of complex coördinates, and let 23, 31, and 12 be perpendicular to 41, 42, 43, respectively. Then

$$(1) \qquad \begin{gathered} (c-b) = ip(a-d), \qquad (a-c) = iq(b-d), \\ (b-a) = ir(c-d), \end{gathered}$$

where p, q, r are real.* Put $\Delta = (c-d)(b-d)(a-d)$. Then

$$(2) \qquad \begin{aligned} A &\equiv (b-c)(c-d)(d-b) = ip\Delta, \quad B \equiv (c-a)(a-d)(d-c) = iq\Delta, \\ C &\equiv (a-b)(b-d)(d-a) = ir\Delta, \quad D \equiv (b-a)(a-c)(c-b) = -ipqr\Delta. \end{aligned}$$

Note that $|A| = 34 \cdot 42 \cdot 23$, $|B| = 41 \cdot 13 \cdot 34$, $|C| = 12 \cdot 24 \cdot 41$, $|D| = 23 \cdot 31 \cdot 12$; and that

$$(3) \qquad A + B + C + D = 0$$

is an identity, being the expansion according to the first row of

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{vmatrix} = 0.$$

Equations (2) and (3) imply

$$(4) \qquad p + q + r - pqr = 0.$$

* It is easy to show that p, q, r are the tangents of the angles of triangle 123, so that (4) is the familiar identity $\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha \tan\beta \tan\gamma = 0$.

Also from (2) it is easy to see that the ratios of A, B, C, D are all real, whence

$$\pm|A| \pm |B| \pm |C| \pm |D| = 0, \tag{5}$$

and, regardless of the algebraic signs of p, q, r, three of the signs in (5) must differ from the fourth.

Conversely, having given (5) with three signs differing from the fourth, comparison with (3) shows that the closed quadrilateral whose sides are parallel and equal in modulus to the quantities A, B, C, D must collapse into a line, whence the ratios of A, B, C, D are all real. If p, q, r be defined as in (1), then $(D/A)(B/C) = -q^2$ is real, so that q is real or imaginary. Then $q, r = q(C/B)$, and $p = q(A/B)$ are alike all real or all imaginary. If they are all imaginary, it is easy to see that the points 1, 2, 3, 4 are collinear; if they are all real, (1) shows that 23, 31, 12 are respectively perpendicular to 41, 42, 43, as required.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

A Textbook of Mathematical Analysis. By R. L. Goodstein. New York, Oxford University Press, 1948. 12+475 pages. $9.00.

The author's confessed aim is to present a logically correct and clear exposition of the differential and integral calculus, which shall still be so simple that it can be "appreciated by any student with a School Certificate knowledge of arithmetic and algebra." He begins with the numbers representable as terminating decimals, and brings in the other real numbers as limits of decimal sequences, that is, as infinite decimals. He is able to develop many of the ideas and theorems concerning infinite sequences and series in his very first chapter of 26 pages. The discussions of continuity and differentiation, introduced in Chapters II and III respectively, are restricted to uniform continuity, and uniform differentiability. Thus every derivative considered is continuous. The indefinite integral (antiderivative) is introduced in Chapter VIII, and numerous formulas are developed in a space of 21 pages. The integral as limit of a sum is discussed in Chapter XIV, with a proof of its existence for uniformly continuous functions. The final chapter XVII on double integrals includes proofs of the equality of repeated integrals, of Green's theorem, and of the formula for transformation of double integrals. There are other chapters on the definition and properties of the elementary transcendental functions, differential geometry of plane curves, differential equations, Taylor's series, and maxima and minima. A proof of the

existence of the limit of a convergent sequence of decimals is given in an appendix. The book closes with 37 pages of examples, of a purely mathematical nature, followed by 101 pages of solutions. A few corrections are listed following the index.

The reviewer does not agree with the view indicated in the author's preface that a presentation of the calculus based to a considerable extent on geometrical intuition necessarily leaves the student "the servant, and never the master, of a fundamental technique." A student's mastery of a subject is a relative matter, and can be increased by a book only if that book attracts the student's interest and stimulates his imagination and intellectual activity. In the reviewer's experience, young students, even those destined to become mathematicians, do not usually begin with an interest in the logical nature of number or the logical proofs of function theory. On the other hand, the budding scientist is usually ready to learn about the tools of mathematical thought, how they work, and what they will do. Thus it would seem that an introduction to the calculus should include correct definitions of the notions of limit, derivative, antiderivative, and the definite integral as the limit of a sum, with ample explanations of the meaning of these notions as well as practice in their use, but without undue emphasis on proofs. Logical proofs and a complete statement of the assumptions used in theorems, should be included when they can be made sufficiently simple. A complete definition of the real number system and derivation of its properties, and other basic logical ideas and proofs of function theory, can be absorbed much more rapidly and effectively by the student after he has had considerable experience with the formal and manipulative side of mathematics.

A natural accompaniment of the representation of the real numbers as decimals is the author's use of the notation $0(n)$ to indicate a number less than $1/10^n$. This notation replaces the ϵ, δ notation in the definitions of limit and continuity, and in fact throughout the book. There are other points at which the terminology is unconventional. This may cause some confusion for the student who pursues mathematics further. The style of printing does not make it easy to pick out the theorems or other significant points—only occasionally is an important result set in bold face. The book seems to be carefully written, and may be useful as supplementary reading for the teacher of calculus even though he is not likely to want it for a class text.

L. M. GRAVES

Essentials of Analytic Geometry. By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Co., 1947. 10+259 pages. $2.80.

This is a revision of the authors' *Analytic Geometry*. Rather extensive changes have been made including the deletion of several chapters and the inclusion of many new exercises. The goal in making the revision was to present the essentials, as the name implies, in a form that could be presented in a four or five semester hour course.

The selection of topics is in general the usual one including a brief treatment

of solid analytic geometry. The latter subject would be more helpful as preparation for the calculus if some treatment were given to solids which are bounded by two or more surfaces. The last chapter, entitled *Systems of Coordinates*, consists of four sections, two on translation and rotation of axes in three space, and two on cylindrical and spherical coordinates. The first two of these sections hardly seem to be "most essential as preparation for the calculus and for engineering courses." Perhaps it is intended that the five pages comprising this chapter might be referred to subsequently in somewhat the same way that a student in this course would refer to the seven pages of formulas from algebra and trigonometry which are included at the end of the book.

There are some other features of the book which deserve comment. One is the introduction of polar coördinates in the first chapter along with rectangular coördinates. Thereafter each topic is discussed in terms of both coördinate systems. As to whether this is desirable or not is probably a matter of taste. Another feature is the lists of supplementary exercises at the ends of most of the chapters. In this connection one wonders why the authors did not go one step further and give references to some other books. Surely, students should be encouraged to use the library.

The authors are to be commended for their attention to accuracy of statement. As examples, on page 4, a distinction is made between a segment and the measure of a segment and on page 24 between the trigonometric functions of angles and of numbers. Figures are well drawn and the book would be quite attractive except for the rather small print in the exercises and examples.

R. H. BARDELL

College Algebra. By T. S. Peterson. New York, Harper and Brothers, 1947. 8+334 pages. $2.50.

In college algebra, the unending controversy between the radicals and the conservatives treats of both the material to be covered and the manner of covering it. The conservative would teach all the traditional topics and drill them thoroughly, in order that the student may become a skilled manipulator able to handle any algebraic difficulty which comes his way. The radical, on the other hand, would select a limited number of basic topics, present them to show their relation to the rest of mathematics, stress algebra as a method of thinking, and try to give unity to the whole.

This book clearly belongs to the former class. It resembles in detail the standard college algebra of the past fifty years. It begins with about 100 pages of review of elementary algebra; it discusses in a minimum of space the underlying principles of each topic and codifies them into Rule 1, Rule 2, · · · ; it has many problems following each article; it covers all the traditional topics from rules of signs to infinite series, with induction, inequalities, Horner's method, the cubic and the quartic, and partial fractions given careful attention. All this is condensed, by means of a concise style and small type, into 298 pages.

Some of the more recent additions to the subject have been included. Sig-

nificant digits are given two pages; interest and annuities rate a 13-page chapter; while statistics and curve fitting occupy the last 15 pages. In this respect the author has been more successful in bringing the book up to date than in the problems, where new cars cost $900, beginning salaries are $1200 per year, cars travel 30 and 24 miles an hour, factory employes earn $2.50 a day and fast trains get up a speed of 60 miles an hour.

The Preface suggests that the book would be useful (1) for students with only one year of algebra behind them; (2) as a terminal course in mathematics; (3) as preparatory for majors in mathematics and science. To the reviewer, it seems (1) too brief, too dull, and too difficult in the first 100 pages for the average student with only one year of algebra; (2) lacking in unity and not sufficiently rich in interpretation and in the breadth of its applications to other fields for a terminal course; and (3) while it might develop skilled manipulators, its problems are so routine and formal that it neglects development of the student's powers of analysis.

However, to a student with sufficient background, with an overwhelming determination to master college algebra, and with an inspiring teacher to provide motivation, this text would surely give a thorough knowledge of the traditional material of college algebra and make him a master of algebraic manipulation.

R. C. HUFFER

Fundamentals of Business Mathematics. By W. R. VanVoorhis and C. W. Topp. New York, Prentice-Hall, 1948. 8+454 pages. $3.75.

Designed to meet the needs of students of business, this book aims to provide the kind and amount of mathematical background needed in the study of finance, commerce, accountancy, business statistics, and related topics. Beginning with a brief review of arithmetic and elementary algebra, the text proceeds to discuss elementary statistical methods, percentage, simple interest and discount, probability, exponents and radicals, logarithms, progressions, problems of finance, and equations. Each of these topics is treated in a simple, clear manner that should appeal to the student. Ample drill problems have been carefully selected and excellent self-tests are given at the end of each chapter.

The reviewer noted but two minor criticisms. On page 9, *factors* are confused with *divisors*. On page 29 it is stated incorrectly (or ambiguously), "This procedure of rounding off the answer to the number of decimal places appearing in the least accurate number entering into the operation is followed in all arithmetic calculations involving approximate numbers."

In the reviewer's opinion, this should prove to be a successful text, particularly in schools and colleges of business administration and in junior colleges.

H. D. LARSEN

Elements of Nomography. By R. D. Douglass and D. P. Adams. New York, McGraw-Hill Book Co., 1947. 9+209 pages. $3.50.

The purpose of this text is to teach the practical construction of elementary

alignment charts, especially to those who study without benefit of instructor. Besides a certain maturity, it presupposes, for the most part, only a knowledge of algebra, plane geometry and logarithms. It should prove satisfactory for class room instruction of engineering students as well as for self-instruction.

For three variables the authors list six types under which most of the charts fall. Type II is the addition nomogram (three parallel straight supports); type III, the multiplication nomogram (two parallel straight supports crossed by a third straight support, the so-called N- or Z-chart); type V, the hexagonal nomogram (three straight supports concurrent in a finite point); type VI, the circular nomogram (one straight support crossing two coincident circular supports); types I, IV are special cases of II, V in which one support bisects, respectively, the distance and the angle between the other two. The main portion of the book thus concerns certain nomograms of genus zero and one of genus two. The final chapter constructs two nomograms of genus one, compounds nomograms of the various types to give nomograms for more than three variables, discusses other related graphical devices and gives charts for solving quadratic, cubic and trinomial equations.

The subject is developed by solving problems illustrative of the various types. The stress is upon mechanical details of the solution, which is on the whole reduced to a series of carefully described steps. Printed forms to be used in solving are given in several cases, with a word of caution against using them before mastering the principles on which they are based. Questions of sign and disposition, which often puzzle the beginner, are dissected at great length. The drawings are large scale, numerous and excellent. The discussion of the technique for the actual drafting is unusually complete.

Although the notation is admirably consistent, it is perhaps excessive and cumbersome. On page 21, for example, ten symbols, two of which involve a subscript on a subscript and also a superscript, are used in the relatively simple operation of adjusting the modulus and the zero of a scale. The exclusion of such tools as analytic geometry and determinants, which have had great heuristic value in the development of the subject and which are available to the advanced undergraduate, handicaps the text for use in a strictly mathematical course as does also the total absence of references to the pertinent literature. The answer to this is perhaps that our overcrowded curriculum has no room for such a course. Since apart from and in spite of its applications nomography is a fascinating branch of mathematics, this state of affairs would indeed be unfortunate, but it would at least justify the authors in limiting their objective, which they seem to have attained with a high measure of success.

J. M. THOMAS

NEW BOOK RECEIVED

Eleven and Fifteen Place Tables of Bessel Functions of the First Kind, to All Significant Orders. By E. Cambi. New York, Dover Publications, 1948. 5+154 pages. $3.95.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1947–48

Pi Mu Epsilon, Louisiana State University

Some of the papers presented at the semi-monthly meetings of *Louisiana Alpha* Chapter of *Pi Mu Epsilon* include:

Some early discoveries in modern mathematics, by Dr. F. A. Rickey

The geometric aspects of the derivative, by Ernest Ikenberry

Approximation of N! for large N, by Dr. D. B. Sumner

Geometric remarks on differential equations, by Dr. Solomon Lefschetz of Princeton University

Mathematics in European universities, by Dr. Valdas Merkys

The Mayan system of numeration, calendar, and astronomy, by M. B. Smith.

Some elementary mathematics used in the study of electronics, by Donald Shipp.

The officers for the year 1947–48 were: President, Wendell Craft; Vice-President, Elinor Ernst; Secretary, Lillian Hudson; Treasurer, M. B. Smith; Faculty Advisor, Prof. H. T. Karnes.

Mathematics Club, Oberlin College

The activities of the *Mathematics Club* of Oberlin College included the following talks:

Pi, by Thomas Morgan

Number systems and the game of Nim, by Emery Thomas

Vectors at sea, by Charles Petree

Men of mathematics, by graphical analysis, by Mary Wright and Jeanne Taylor

An infinity of paradoxes, by Ruth Berger

Number theorems, by Lester Arnold

The foundations of geometry, by Joe Solomon

Ballistic pendulums, by Bruce Clark

Plane surveying, by Robert Keesey

Modern mathematics, by Walter Wood.

Prof. S. W. McCuskey of Case Institute of Technology spoke on the subject *Astronomical statistics* at the annual club banquet.

Officers for 1947–48 were: President, Lester Arnold; Vice-president, Ruth Berger; Secretary-treasurer, Mary Wright; Publicity chairman, Evelyn Schmidt; Social Chairman, Rosalind Monastersky; Faculty Advisor, Prof. E. P. Vance.

Officers for 1948–49 are: President, Emery Thomas; Vice-President, Joe Solomon; Secretary-treasurer, Evelyn Schmidt; Publicity Chairman, Ann Johnson; Social Chairman, Jeanne Taylor; Faculty Advisor, Prof. R. W. Wagner.

Mathematics Club, Sampson College

The Mathematics Club of Sampson College functions to promote interest in mathematics and in its application. Aiming for this goal, the club presented the following lectures:

Dimensional analysis, by Mr. Petrie
Mathematics and chemistry, by Jack Bulloff
What is a real number?, by Mr. R. G. Albert
Ballistics, by Edwin Peck
Mathematics in music, by Dr. Paul Squires
Cartography, by Isidore Greenberg
Plane geometry and the nine-point circle, by Edmund Frankel
Vector analysis, by Winton Laubach
Mathematics in mechanical optics, by H. C. Fish
Magic, by Jack Martin
Mathematics of tool and die making, by William Krasnow.

An intra-club mathematical contest was held during one of the meetings.

The novel event of the year was the presentation of Mack Martin's play, *How to catch a tortoise*, enacted by Eugene Hochdanner, Otto Fabian, Jack Martin and Howard Fish. The core of the play was a discussion between a modern philosopher and Zeno (returned to earth) upon the paradoxical nature of motion.

The schedule of future club activities includes a treasure hunt in mathematics books, a mathematics exhibition, and a mathematics party. Five minute reports by students on books in mathematics will be featured in future meetings.

The Mathematics Club boasts its own library room with blackboard, books and periodicals for informal study. Plans for a mathematical mural and a cabinet of geometrical models have been initiated.

The members elected to officers' positions at the second meeting were: President, Eugene Hochdanner; Vice-president, Robert Forrest; Secretary, Howard Fish; Publicity Director, Gilbert Friedenreich; Science Council Representative, A. W. Vandewinckel; Program Committee, William Krasnow and Arthur Siegel; Faculty Advisor, Mr. Richmond G. Albert.

Mathematics-Physics Club, College of Saint Teresa

Applications of mathematics to various professions was the theme of the year's programs of the Mathematics-Physics Club of College of Saint Teresa. Guest speakers devoted evenings to architecture, engineering, biophysics, and chess. Members of the club gave talks on *Bridges*, *Body mechanics in nursing arts*, *The sextant*, and *Celestial navigation*. Mathematics films also had their part in the programs. The year closed with a mathematical picnic at which the year's theme was put into effect in a recreational way.

Officers for the years were: President, Betty Gervais; Vice-president, Genevieve Suprenant; Secretary, Mary Jane Dyer; Treasurer, Joan Koch.

Kappa Mu Epsilon, Upsala College

The papers presented at the regular monthly meetings of the *New Jersey Alpha* Chapter of *Kappa Mu Epsilon* were:

Mathematics at Shrivenham University and other English Colleges, by Dr. D. R. Davis of Montclair State Teachers College

An introduction to nomography, by Marjorie Cohen

Different methods of proving the Pythagorean theorem, by Frances Rischmuller.

Nomographic chart for quadratic equations, by Robert Wallace

The body of mathematics as we have it today, by Dr. M. A. Nordgaard, faculty advisor

Relationship of mathematics to philosophy, by June Davidson

Theory of probability, by James Gill and Martin Monroe.

At the Ninth Annual Banquet Dr. Nathan Lazar of Teachers College, Columbia University, spoke on *The development of the abacus*. The members of the fraternity joined the Mathematics Club in a trip to Hayden Planetarium.

The officers during the year 1947–48 were: President, Marjorie Cohen; Vice-President and Treasurer, June Davidson; Recording Secretary, Frances Rischmuller; Corresponding Secretary, Dr. M. A. Nordgaard.

The officers for the year 1948–49 are: President, Frances Rischmuller; Vice-President, Robert Wallace; Recording Secretary, James Gill; Treasurer, Martin Monroe; Corresponding Secretary, Dr. M. A. Nordgaard.

Mathematics Club, Montana State University

The *Mathematics Club* of Montana State University held ten regular meetings during the year 1947–48. Faculty members of the mathematics and physics departments and undergraduate students presented the following papers:

Diophantine equations, by Mr. Coffee

Atomic research, by Dr. Miller

History of pi, by Don Marshall

Probability and games of chance, by Dr. Ostrom

Curve fitting, by Dr. A. S. Merrill

Mathematics in occupations, by Frank McElwain

One to one correspondence, by Dr. Lennes

Rubber sheet geometry, by Paul Rygg.

There were two social meetings during the year, one at the home of Dr. and Mrs. A. S. Merrill during the winter quarter, and the other was the annual Mathematics-Chemistry Club picnic in the spring.

The officers for 1947–48 were: President, Don Marshall; Vice-President, Jean Popham; Secretary-Treasurer, Veronica Kreitel; Faculty advisor, Dr. T. G. Ostrom.

The officers for 1948–49 are: President, Paul Rygg; Vice-President, Veronica Kreitel; Secretary-Treasurer, Elsie Taylor; Faculty advisor, Dr. T. G. Ostrom.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: C. B. Allendoerfer, Norman Anning, W. L. Ayres, E. F. Beckenbach, L. M. Blumenthal, H. W. Brinkmann, W. E. Byrne, G. F. Carrier, E. W. Chittenden, Paul Civin, A. B. Coble, Nathaniel Coburn, C. J. Coe, Richard Cohn, Richard Courant, H. S. M. Coxeter, H. T. Davis, Douglas Derry, P. S. Dwyer, H. T. Engstrom, P. Erdös, H. W. Eves, L. R. Ford, Tomlinson Fort, J. S. Frame, Orrin Frink, W. H. Gage, J. W. Givens, L. M. Graves, N. G. Gunderson, F. S. Harper, G. E. Hay, E. R. Heineman, M. R. Hestenes, T. H. Hildebrandt, R. T. Hood, G. B. Huff, H. K. Hughes, R. D. James, Fritz John, L. S. Johnston, P. S. Jones, Fred Kiokemeister, H. D. Larsen, D. H. Lehmer, J. N. Livingood, C. C. MacDuffee, Saunders MacLane, H. B. Mann, W. T. Martin, N. H. McCoy, A. D. Michal, Norman Miller, C. N. Moore, Marston Morse, Ivan Niven, Rufus Oldenburger, C. D. Olds, Oystein Ore, C. G. Phipps, George Pólya, G. B. Price, E. D. Rainville, O. J. Ramler, J. F. Randolph, W. T. Reid, P. R. Rider, G. de B. Robinson, J. B. Rosser, I. M. Sheffer, H. L. Smith, C. E. Springer, E. P. Starke, J. L. Synge, A. H. Taub, R. M. Thrall, R. S. Underwood, H. S. Vandiver, R. J. Walker, J. L. Walsh, S. W. Warschawski, Louis Weisner, Charles Wexler, R. L. Wilder, H. E. Wolfe, M. A. Zorn.

PREDOCTORAL FELLOWSHIP OF SIGMA DELTA EPSILON

The Fellowship Board of Sigma Delta Epsilon, Graduate Women's Scientific Fraternity, announces that applications for the Sigma Delta Epsilon Research Fellowship in Science should be submitted before February 1, 1949. Application blanks may be secured from Dr. Virginia Barton, 7 Chemistry Annex, University of Illinois, Urbana, Illinois. Women of outstanding ability, who are predoctoral candidates in the mathematical, physical, or biological sciences and who need financial assistance to further a well-defined research project, are eligible candidates.

HARRY BATEMAN RESEARCH FELLOWSHIPS

Applications are invited for these post-doctorate fellowships in Pure Mathematics at the California Institute of Technology. Recipients will devote the major part of their time to research. In addition, they are expected to teach one upperclass course in mathematics. The stipend is $3600 for the academic year. For application blanks and further information, address the Dean of the Faculty, California Institute of Technology, Pasadena 4, California. Applications must be returned before March 1, 1949.

RESEARCH GRANTS OF THE INSTITUTE FOR ADVANCED STUDY

The School of Mathematics, Institute for Advanced Study, will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and do research work at Princeton during the academic year 1949–50. Candidates must have given evidence of ability in research comparable at least with that expected for the Ph.D. degree. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, New Jersey and are returnable by February 1, 1949.

ORGANIZED RESERVE RESEARCH GROUPS OF THE UNITED STATES ARMY

The Department of the Army has established a program of particular interest to mathematicians and other scientists, who hold Reserve commissions in the Army, and who are professionally engaged in teaching or research and development. The objectives of the program are to: (1) maintain the useful affiliation of mathematicians and other scientists with the Organized Reserve Corps, (2) provide peacetime Reserve assignments for these officers, enabling optimum utilization of their education, experience and skills, (3) furnish mobilization assignments which will fully utilize their talents, and (4) adequately prepare these officers for mobilization.

Reserve officers who are currently engaged in civilian research, college or university teaching, or industrial research or development, or who in the past have had specific research experience are eligible to make application for assignment to an Organized Reserve Research and Development Group. A group may be organized in any locality where there are twenty or more qualified officer scientists who desire to participate in the program. A subgroup may be organized with ten qualified members.

Inquiry about organization of an Organized Reserve Research and Development Group or about assignment to a group already organized should be made of the Unit Instructor, ORC, or of the Senior Army Instructor, ORC, in the locality in which the officer resides.

PERSONAL ITEMS

Professor Charles Blanc of the University of Lausanne and Professor Albert Pfluger of the Swiss Federal Institute of Technology have been elected President and Vice-President, respectively, of the Swiss Mathematical Society.

Professor L. M. Graves of the University of Chicago has received the President's Certificate of Merit for his work on the panel of applied mathematics of the National Defense Research Committee.

Associate Professor E. H. C. Hildebrandt has been elected President of the National Council of Teachers of Mathematics. He is also National Secretary-Treasurer of Pi Mu Epsilon and Secretary of the Mathematics Section of the Central Association of Science and Mathematics Teachers.

Professor George Polya of Stanford University received an honorary degree

from the Swiss Federal Institute of Technology, Zurich, on November 15, 1947.

Professor Hassler Whitney of Harvard University has received an honorary degree of Doctor of Science from Yale University.

It has been announced by the U. S. Army that the following mathematicians will serve as an advisory committee especially with regard to speed calculation: Dr. Hendrik Bode, Bell Telephone Laboratories; Professor Richard Courant, New York University; Professor John von Neumann, Institute for Advanced Study; Professor J. B. Rosser, Cornell University; Professor H. P. Robertson, California Institute of Technology; Professor J. J. Stoker, New York University; Professor E. R. Lorch of Columbia University will be mathematics adviser to the research group.

Alabama Polytechnic Institute announces the promotions of Associate Professor Ernest Williams to a professorship and Assistant Professor S. L. Thompson to an associate professorship.

Arizona State College reports: Associate Professor R. B. Lyon, who is on sabbatical leave, is at the University of Texas; Dr. Lee Byrne, formerly of Purdue University, has been appointed Special Lecturer in Mathematics.

At Boston College, Dr. R. J. Marcou has been promoted to a professorship; Dr. H. G. Haefeli and Dr. T. S. Motzkin have been appointed to associate professorships.

Butler University makes the following announcements: Professor H. E. Crull, head of the Department of Mathematics, is now also Director of the University College; Mr. William Fuller and Miss Jane Uhrhan have been appointed to instructorships.

Colorado State College of Education announces the appointment of Mr. K. E. Hansen to an instructorship. A very successful Summer Conference on Science, Mathematics and Social Studies was held on July 9–10, 1948.

At De Paul University, Mr. Chester Pachuki and Mr. Arthur Saastad have been appointed to instructorships.

Drake University announces the promotions of Associate Professor B. E. Gillam to a professorship and Instructor R. W. Gardner to an assistant professorship.

Duke University announces the appointments of the following instructors: J. R. Garrett, M. P. Jarnagin, T. D. Reynolds, all formerly graduate students at Duke University.

Emory University reports: Professor C. G. Latimer has been appointed Chairman of the Department of Mathematics; Mr. C. R. Partington of Purdue University has been appointed to an instructorship.

Harvard University makes the following announcements: Assistant Professor George Mackey has been promoted to an associate professorship; Dr. Oscar Goldman of Princeton University, Dr. John Gurland of the University of California in Berkeley and Dr. Raymond Redheffer of the Massachusetts Institute of Technology have been appointed Benjamin Peirce Instructors of Mathematics; Professor D. V. Widder is on sabbatical leave during 1948–49 and is spending

the year in California. Professor Garrett Birkhoff was abroad during the spring term and summer as a Guggenheim Fellow; he lectured in the London Mathematical Society and at the Institut Henri Poincaré. Professor Mackey delivered an invited address at a conference held at the University of Chicago on Topological Groups during the spring of 1948.

Johns Hopkins University announces: Dr. W. L. Chow of the University of Shanghai and the Institute for Advanced Study has been appointed to an associate professorship; Dr. L. G. Peck of New York University and the Institute for Advanced Study has been appointed Assistant Professor; Mr. E. D. Carey of New York University, Dr. M. L. Juncosa of Cornell University, and Doctors E. A. Coddington, C. R. Putnam, Sylvan Wallach of Johns Hopkins University have been appointed to instructorships. Professor F. D. Murnaghan has resigned to accept an offer from the Brazilian government; he will assume his new duties in Brazil in January 1949. Professor B. L. van der Waerden has resigned to resume his duties with the Royal Dutch Oil Company.

Lehigh University reports the following: Associate Professor A. E. Pitcher has been promoted to a professorship; Mr. Paul Meier of Princeton University has been appointed to an assistant professorship; Mr. Michael Tikson has been appointed to a graduate assistantship; Assistant Professor H. G. Means has retired.

Louisiana State University makes the following announcements: Assistant Professor L. I. Wade of Duke University has been appointed Professor and Head of the Department of Mathematics; Mr. B. B. Townsend of the University of Texas has been appointed to an assistant professorship; Professor I. C. Nichols has retired with the title of Professor Emeritus.

New York University announces these promotions: Associate Professor H. R. Cooley to a professorship; Assistant Professor H. E. Wahlert to an associate professorship; Dr. Joseph Keller who has been connected with the Institute of Mathematics and Mechanics to a research assistant professorship; Instructors Beatrice G. Edison and David Gans to assistant professorships. The Pi Mu Epsilon Mathematics Contest was held at Washington Square College on March 13, 1948 with representatives of one hundred eighty-three schools participating.

Northwestern University reports the following: Instructor D. H. Potts has been promoted to an assistant professorship; Mr. E. W. Banhagel of Wayne University, Mrs. Helen Betz, Morton Junior College, and Mr. W. M. Boothby, Rackham Fellow at University of Michigan, have been appointed to instructorships; Professor W. T. Reid served as visiting professor at the University of California at Los Angeles during the second Summer Session of 1948. Assistant Professor M. E. Wescott has been elected Chairman of Board of Editors of Industrial Quality Control.

At Purdue University the following appointments to instructorships have been made: Mrs. Nadine Johannesen of Wayne University, Miss Ruth King and Mr. James McKnight of Purdue University, Dr. Harold Shniad, University of

California at Los Angeles, and Mr. Erskine St. Clair, University of Michigan.

Rice Institute announces the promotion of Dr. H. D. Brunk to an assistant professorship.

Sam Houston State Teachers College reports: Mr. M. W. Wells of Texas Agricultural and Mechanical College has been appointed to an assistant professorship; Associate Professor B. M. Wall is on leave and is studying at the University of Texas.

At Simmons College a separate Department of Mathematics has been organized effective with this academic year. Miss Sybil Tanenbaum, formerly at Brown University, has been appointed to an instructorship.

Syracuse University announces the following: Assistant Professor F. W. Borgward and Professor I. S. Carroll have retired with the title of Professor Emeritus; Dr. O. O. Pardee has been appointed to an instructorship. Professor P. C. Rosenbloom has returned from Sweden where he spent eighteen months on a Guggenheim fellowship. Professor Abe Gelbart has returned after a year at the Institute for Advanced Study.

United States Military Academy makes the following announcements: Colonel W. W. Bessell, Jr. has been appointed Professor and Head of the Department of Mathematics; Colonel C. P. Nicholas has been appointed to a professorship; Lieutenant Colonel R. C. Yates has been promoted to an associate professorship; Lieutenant Colonel J. T. Honeycutt and Lieutenant Colonel A. W. Oberbeck have been promoted to assistant professorships. Brigadier General Harris Jones, formerly head of the Department of Mathematics, is now Dean of Academic Board; Colonel W. N. Underwood is District Engineer on Guam; Lieutenant Colonel J. S. B. Dick is overseas.

United States Naval Academy announces the following promotions: Assistant Professors J. M. Holme, J. F. Milos, R. C. Morrow, V. N. Robinson, S. S. Saslaw, and C. W. Seekins to associate professorships; Instructors B. H. Bulkstra, A. R. Craw, M. V. Gibbons, J. R. Gorman, E. C. Gras, F. P. Kowalewski, K. F. McLaughlin, G. J. Mann, Joseph Milkman, Robert Molly, N. O. Niles, A. J. Pejsa, J. W. Popow, R. W. Rector, R. C. Simpson, H. K. Sohl, A. H. Streinbrenner, M. F. Stilwell, W. J. Strange, G. R. Strohl, Jr., E. G. Swafford, C. E. Thompson, J. A. Tierney, E. C. Watters, Jr., J. H. White, and H. Wierenga to assistant professorships.

At the United States Naval Postgraduate School, Assistant Professor R C. Campbell of the University of Pennsylvania and Assistant Professor B. J. Lockhart of the University of Michigan have been appointed to assistant professorships; Mr. N. F. Spraggins, formerly at Stanford University, has been appointed to an instructorship.

University of Chicago reports: Visiting Professor Andre Weil has been promoted to a professorship; Assistant Professor Leopoldo Nachbin of the University of Brazil and Professor Candido Lima Da Silva Dias of the University of Sao Paulo have been appointed Research Associates. Instructor E. J. Akutowicz is now an Atomic Energy Post Doctoral Fellow; Instructor J. B. Crabtree is with

the ONR in London.

University of Colorado announces the appointment of Mr. Burrowes Hunt, Jr. and Mr. Gideon Culpepper of the University of Colorado to instructorships.

University of Florida announces: Mr. Kenneth Walters of Lehigh University has been appointed to an instructorship; Mr. L. H. Potter, graduate student of the University of Florida, has been appointed to an acting instructorship; Mr. M. J. Cleveland, Mr. A. J. Owens, Mr. J. M. Robertson, and Mr. W. B. Stovall, Jr. have been appointed to teaching assistantships; Mr. J. A. Adkinson, Mr. J. R. Duffett, Mr. G. B. Findley, and Mrs. Orpha Ingwalson have been appointed to graduate assistantships.

University of Georgia makes the following announcements: Professor Tomlinson Fort has been appointed Distinguished Service Professor; Associate Professor J. A. Ward was promoted to a professorship; Professors W. V. Parker and J. A. Ward have been given special research grants by the University of Georgia; new members of the staff are Associate Professor G. M. Conwell and Instructor James Bercos; Professor G. B. Huff is on leave of absence for the current year and is at Harvard University. The Department of Mathematics has been authorized to offer the Ph.D. Degree.

At the University of Houston, Associate Professor A. A. Aucoin has been promoted to a professorship and Assistant Professor Albert Newhouse has been promoted to an associate professorship.

University of Kansas reports: Associate Professor P. O. Bell has been promoted to a professorship; Dr. I. N. Herstein, Dr. G. K. Overholtzer, Mr. Robert Marceau, and Miss Minnie Stewart have been appointed to instructorships; Associate Professor H. E. Jordan has retired with the title of Associate Professor Emeritus. Mr. S. G. Kneale, graduate student, has been awarded a scholarship at Harvard University.

The University of Illinois announces: Professor W. G. Madow of the University of North Carolina has been appointed Professor of Mathematical Statistics; Professor Loo Keng Hua, Institute for Advanced Study, has been appointed Visiting Professor; Dr. Lowell Schoenfeld, Harvard University, and Dr. Irving Reiner, formerly of the Institute for Advanced Study, have been appointed to assistant professorships; Assistant Professor F. E. Hohn of the University of Maine, Mr. R. G. Langebartel of the University of Illinois, and Mrs. Irma M. Reiner of Temple University have been appointed to instructorships; Instructors W. A. Ferguson, Joseph Landin, B. E. Meserve, and M. E. Munroe have been promoted to assistant professorships.

Dr. H. L. Alder, instructor at the University of California at Berkeley, is now an instructor at the University of California at Davis.

Associate Professor H. C. Ayres of the United States Naval Academy has accepted a position as professor and head of the Department of Mathematics of Jersey City Junior College.

Dr. L. J. Burton has been appointed to an assistant professorship at Bryn Mawr College.

Mr. P. F. Cauffman of Lehigh University has been appointed Professor and Head of the Department of Mathematics of State Teachers College, Salisbury, Maryland.

Dr. J. H. Curtiss, chief of the National Applied Mathematics Laboratories, has assumed temporary additional duties as acting chief of the Institute for Numerical Analysis at the University of California at Los Angeles.

Assistant Professor M. P. Fobes of the College of Wooster has been promoted to the headship of the Department of Mathematics.

Associate Professor T. J. Higgins of the Illinois Institute of Technology has been appointed to a professorship in the Department of Electrical Engineering of the University of Wisconsin.

Mr. K. R. Jones, who has been associated with the NEPA Project, Oak Ridge, is now employed at Ordnance Research Number One, University of Chicago.

Dr. Samuel Karlin has been appointed Research Instructor for the current academic year at California Institute of Technology.

Mr. A. W. Kaufman is now an instructor at Aurora College.

Assistant Professor R. R. Kuebler of Dickinson College has been promoted to an associate professorship.

Mr. J. V. Limpert of Syracuse University has been appointed to an assistant professorship at St. Lawrence University.

Dr. Eugene Lukacs has a position as statistician at the United States Naval Ordnance Test Station, Inyokern, California.

Mr. K. E. McLachlan of Baylor University has been promoted to an assistant professorship.

Mr. H. W. Morrow, Jr. has been appointed Instructor in Mechanics at the University of Florida.

Mr. L. R. Norwood of Yale University is now employed as a mathematician at the United States Army Signal Corps Laboratory, Fort Monmouth, New Jersey.

Miss Gloria Olive of the University of Arizona has accepted an appointment as instructor at Idaho State College.

Mr. R. V. Person has been appointed to an instructorship at American University.

Reverend Albeni Poitras of St. Joseph's University, New Brunswick, Canada has been made Dean of Studies.

Mr. R. M. Schmied, graduate fellow at Brown University, has been appointed to an instructorship at Tulane University.

Assistant Professor Abraham Schwartz of Pennsylvania State College is now an instructor at the College of the City of New York.

Mr. R. R. Seeber has been made Senior Staff Member with the International Business Machines Corporation, New York City.

Mr. W. H. Simons of the University of British Columbia has been promoted to an assistant professorship.

Mr. M. L. Stein of the University of California at Los Angeles has accepted a position as mathematician at the Institute for Numerical Analysis, U. C. L. A.

Mr. H. W. Stephens of the University of Florida has accepted an instructorship at the University of Maryland.

Professor Gabor Szego of Stanford University was on leave of absence during the Summer and Autumn Quarters for scientific work in Europe.

Associate Professor Alexander Tartler, Drexel Institute of Technology, has been promoted to a professorship.

Mrs. Maria Weber has been appointed to an instructorship at Goucher College.

Professor C. O. Williamson of the College of Wooster has been promoted to the headship of the Department of Applied Mathematics.

Mr. H. W. Zeoli, formerly a secondary school teacher at Hyannis, Massachusetts, has been appointed to an assistant professorship at Central Michigan College.

Professor Emeritus G. W. Finley of Colorado State College of Education died on May 17, 1948.

Professor Emeritus J. H. M. Wedderburn of Princeton University died early in October, 1948.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-six persons have been elected to membership on applications duly certified:

M. K. Al-Ghita, M.S. (Michigan) Graduate Student, University of Michigan, Ann Arbor, Mich.

H. W. Baeumler, B.S. (Buffalo S.T.C.) Instructor, University of Buffalo, N. Y.

D. Y. Barrer, B.S. (New York University) Graduate Asst., Northwestern University, Evanston, Ill.

R. C. F. Bartels, Ph.D. (Wisconsin) Asst. Professor, University of Michigan, Ann Arbor, Mich.

Rene Baus, Jr., B.S. (Tulane) Graduate Student, Tulane University, New Orleans, La.

Mildred E. Blackman, B.S. (St. Ambrose) Instructor, St. Ambrose College, Davenport, Iowa

May C. Blackstock, M.A. (Brown) Instructor, University of Tennessee, Knoxville, Tenn.

Daniel Block, B.A. (Yeshiva) Instructor, Yeshiva University, New York, N. Y.

W. H. Blum, M.A. (West Virginia) 505 Kruger Street, Elm Grove, W. Va.

Mamie I. Braswell, M.A. (George Peabody) Asso. Professor, Alabama College for Women, Montevallo, Ala.

R. C. Brown, Jr., M.S. (West Virginia) Instructor, West Virginia University, Morgantown, W. Va.

R. K. Brown, B.Sc. (Muhlenberg) Teaching Asst., Rutgers University, New Brunswick, N. J.

EMALOU BRUMFIELD, M.A. (Ohio State) Instructor, Kent State University, Kent, Ohio

J. M. CAMERON, M.S. (North Carolina State College) Mathematician, National Bureau of Standards, Washington, D. C.

W. F. CARNES, A.M. (Harvard) Tutor, St. John's College, Annapolis, Md.

J. G. CHRISTIANO, M.S. (Pittsburgh) Asst. Professor, University of Pittsburgh, Pa.

J. L. COLLINS, B.S. (St. Mary's) Instructor, St. Ambrose College, Davenport, Iowa

BROTHER DAMIAN CONNELLY, F.S.C., Ph.D. (Catholic University) Asst. Professor, LaSalle College, Philadelphia, Pa.

R. R. CROXTON, M.Ed. (South Carolina) Instructor, University of South Carolina, Columbia, S. C.

R. A. DEUTSCH, B.S. (Long Island) Electrical Engineer, Kellex Corporation, New York, N. Y.

P. W. EDMONSON, M.A. (Michigan) Teacher, Dearborn Junior College, Mich.

C. G. GOA, D.Ing. (Venezuela) Engineer, Ministry of Public Works, Caracas, Venezuela

RUTH E. GOODMAN, Ph.D. (Pennsylvania) Asst. Professor, Duquesne University, Pittsburgh, Pa.

RUBY M. GRIMES, A.M. (Illinois) Asst. Professor, North Dakota Agricultural College, Fargo, N. D.

G. B. HAGGERTY, M.A. (Bucknell) Asst. Professor, Rhode Island State College, Kingston, R. I.

EXA O'D. HARDIN, B.S. (Prairie View College) Teacher, Phillis Wheatley High School, Houston, Tex.

R. C. HASELTINE, M.A. (Pennsylvania State College) Chairman of Dept., Pennsylvania State College (Swarthmore Center), Swarthmore, Pa.

HELEN M. HEATER, M.S. (West Virginia) Instructor, West Virginia University, Morgantown, W. Va.

P. F. HULTQUIST, B.A. (Colorado) Assistant, University of Wisconsin, Madison, Wis.

R. Y. IWANCHUK, M.A. (Columbia) Asst. Professor, Kent State University, Kent, Ohio

AYRLENE MCG. JONES, M.A. (Texas) Asst. Professor, Mississippi Southern College Hattiesburg, Miss.

H. J. JONES, M.A. (Texas) Teacher, Leuzinger High School, Lawndale, Calif.

J. J. JONES, M.A. (George Peabody) Asst. Professor, Mississippi Southern College, Hattiesburg, Miss.

R. L. JONES, M.S. (South Carolina) Adjunct Professor, University of South Carolina, Columbia, S. C.

ABRAHAM KARRASS, Boys High School, Brooklyn, N. Y.

F. L. KIOKEMEISTER, Ph.D. (Wisconsin) Asso. Professor, Mount Holyoke College, S. Hadley, Mass.

E. H. LARGUIER, S.J., Ph.D. (Michigan) Professor, Spring Hill College, Mobile, Ala.

J. A. LA RUE, B.A. (West Virginia) Teaching Fellow, West Virginia University, Morgantown, W. Va.

T. H. LEE, M.A. (North Carolina) Adjunct Professor, University of South Carolina, Columbia, S. C.

E. J. LOWRY, M.Sc. (Nebraska) Professor, Hastings College, Neb.

W. C. LOWRY, M.Ed. (Ohio University) Instructor, Kent State University, Kent, Ohio

L. J. MATTESON, JR., A.B. (Colgate) Actuarial Trainee, Mutual Life Insurance Co., New York, N. Y.

J. E. MCGAUGHY, B.S. (Chicago) Instructor, Lawrence College, Appleton, Wis.

J. S. MINAS, Student, Wayne University, Detroit, Mich.

B. N. MOYLS, Ph.D. (Harvard) Asst. Professor, University of British Columbia, Vancouver, B. C.

T. F. MULCRONE, S.J., M.S. (Catholic University) Asst. Professor, Spring Hill College, Mobile, Ala.

BERNICE ORSHANSKY, B.A. (Hunter) Mathematics Editor, Educational Testing Service, Cooperative Test Division, New York, N. Y.

S. J. PAGANO, M.A. (Washington University) Instructor, Missouri School of Mines, Rolla, Mo.

L. G. RIGGS, M.S. (Syracuse) Instructor, Northwestern University, Evanston, Ill.

G. B. ROBISON, M.A. (Columbia) Asst. Professor, Sampson College, N. Y.

REV. E. J. ROCHE, M.S. (Notre Dame) Professor, St. Dunstan's University, Charlottetown, P. E. I.

W. G. ROULEAU, A.B. (Catholic University) Statistical Clerk, National Bureau of Standards, Washington, D. C.

R. R. SEEBER, JR., A.B. (Harvard) Senior Staff Member, International Business Machine Corp., New York, N. Y.

C. J. SENSALE, Student, New Jersey State Teachers College, Montclair, N. J.

J. R. SEWELL, B.S. (Rollins) Graduate Student, Northwestern University, Evanston, Ill.

E. I. SHAPIRO, M.A. (Cornell) Instructor, Brooklyn College, N. Y.

R. L. SHIVELY, M.S. (Michigan) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

C. E SHOTWELL, B.A. (Tusculum) Asst. Instructor, University of Missouri, Columbia, Mo.

A. T. SKINNER, M.A. (Columbia) Asst. Professor, Sampson College, N. Y.

R. E. SMART, B.S. (Ohio State) Engineer, Anchor Hocking Glass Corporation, Lancaster, Ohio

J. L. SOLOMON, Student, Oberlin College, Ohio

R. L. SPENCER, M.A. (Michigan) Instructor, Dearborn Junior College, Mich.

W. L. STAMEY, B.A. (Colorado State) Asst. Instructor, University of Missouri, Columbia, Mo.

R. H. STARK, Ph.D. (Northwestern) Instructor, Northwestern University, Evanston, Ill.

R. F. STEWARD, B.S. (Wheaton) Teaching Asst., New Jersey College for Women, New Brunswick, N. J.

R. B. STILES, A.B. (Middlebury) Instructor, Vanderbilt University, Nashville, Tenn.

J. C. THURMAN, M.A. (Vanderbilt) Asst. Professor, Vanderbilt University, Nashville, Tenn.

J. T. VALLANDINGHAM, A.B. (Georgetown) Instructor, Cumberland College, Williamsburg, Ky.

JOHNNIE VAN, P.O. Box 27, Hastings, Fla.

F. E. VAN BERGEN, D.Sc. (Brussels) Teacher, Royal Athenaeum of St. Niklaas, St. Niklaas-Waas, Belgium

A. A. VUYLSTEKE, Student, Wayne University, Detroit, Mich.

J. G. WENDEL, Ph.D. (California Tech.) Instructor, Yale University, New Haven, Conn.

W. D. WILLIAMS, M.S. (Illinois) Instructor, Hannibal La Grange College, Hannibal, Mo.

E. H. WOHLER, M.A. (Toledo) Asst. Professor, Bowling Green State University, Ohio

J. E. WOOD, M.A. (Colorado State) Director of Aviation, Scottsbluff Junior College, Neb.

L. G. WORTHINGTON, M.A. (North Texas State) Head of Department, John Tarleton College, Stephenville, Tex.

THE APRIL MEETING OF THE KANSAS SECTION

The thirty-second annual meeting of the Kansas Section of the Mathematical Association of America was held at Mount St. Scholastica College in Atchison, on Saturday, April 10, 1948. Sessions were held in the morning and afternoon. Sister Helen Sullivan presided at these sessions.

The attendance was one hundred forty-one including the following forty-four members of the Association: R. W. Babcock, Wealthy Babcock, H. H. Barnett, Florence Black, Frances Breneman, C. H. Brown, Virginia L. Chatelain, L. E. Curfman, Lucy T. Dougherty, Paul Eberhart, Albert Furman, W. H. Garrett, Laura Z. Greene, Edison Greer, J. R. Hanna, W. C. Janes, L. E. Laird, C. F. Lewis, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, C. V. Newsom, Sister Jeanette Obrist, S. T. Parker, O. J. Peterson, Rev. P. S. Pretz, G. B. Price, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, E. S. Robbins, R. G. Sanger, Robert Schatten, J. A. G. Shirk, G. W. Smith, R. G. Smith, Sister Helen Sullivan, C. B. Tucker, Gilbert Ulmer, Frances E. Walsh, E. B. Wedel, A. E. White, P. M. Young.

At the business meeting the following officers were elected for next year: Chairman, R. G. Sanger, Kansas State College; Vice-Chairman, R. G. Smith, Kansas State Teachers College, Pittsburg; Secretary-Treasurer, Anna Marm, Bethany College.

The following papers were presented:

1. *Establishing our mathematical perspective*, by Professor C. V. Newsom, Oberlin College.

A fundamental problem of education is that of perfecting the powers of a person, especially insofar as he is enabled to comprehend and control his environment. This requires an understanding of the methods employed in attempting to bridge the gap between the mind of man and the conduct of nature. The systematic construction of various models, including symbolic systems, that are useful in coordinating and predicting events in nature characterized the scientific reformation of the seventeenth century. Since that time the use of the symbolic patterns of mathematics has grown, and virtually every science acknowledges the importance of mathematical procedures. These facts make it essential that early courses in mathematics should make clear the true role of mathematics. It should be emphasized that (a) mathematics is manmade, and, in general, a mathematical pattern fits any segment of nature only approximately, although the fit may be amazingly good; (b) the utility of mathematics would be quite limited except for the generality inherent in its formalizations; (c) abstract mathematical systems employ the method of postulational thinking, and every attempt should be made to acquaint students with the mathematical method and the desire for rigor. A corollary of these conditions is the fact that the mathematician, or at least the teacher of mathematics, must be extremely versatile in his preparation and interests.

2. *Panel: Mathematics and the development of civilization.*

This discussion consisted of five parts. First the role of mathematics in physical sciences was treated by G. B. Price of the University of Kansas. He remarked that mathematics is concerned with the study of structure. The role of mathematics in the physical sciences is to supply, through the study of the structure of mathematical systems, mathematical models which can be used to explain physical phenomena and to predict results which have not been observed. Three important examples are Newton's model of the universe, Fourier's model for the study of the flow of heat, and Einstein's model of the universe.

Then Frances M. Breneman of Washburn University dealt with the role of mathematics in the business world. Direct application of mathematics to accounting, interest, and annuities was mentioned. The place of mathematics in a liberal arts education and the importance of this training in the business world was considered. Emphasis was placed upon the importance of sound reasoning and the role of formal mathematics in developing this type of thinking.

Mathematics in its relation to learning was discussed by Frances E. Walsh of St. Louis University. It was remarked that the title of this paper might well be *A Defense of the Mathematics Scholar*. The author advocates an attempt to elevate the masses to accept the criteria of the scholar. This necessitates the creation of scholars among whom the mathematician holds a prominent position because he deals with a speculative science. Intrinsically mathematics has much to offer the scholar. In considering the study of pure mathematics the following points illustrate the contribution of mathematics in the training of scholars: (1) The study of mathematics from a historical point of view gives one a cultural background; (2) The pursuit of mathematics gives the student an understanding and an appreciation of what constitutes rigorous thinking; (3) The process of deductive reasoning provides training in the ability to relate precise statements so as to formulate valid conclusions.

O. J. Peterson of Kansas State Teachers College at Emporia then discussed the general attitudes to be developed in our future mathematics teachers. He presented a number of desirable general attitudes consistent with present-day educational aims, and indicated how these attitudes would aid in promoting general citizenship competency and professional or vocational success.

The merits and limitations of the mathematical method were then treated by P. M. Young of Kansas State College. Abstraction, generalization, logical structure and rigor were discussed as merits of the mathematical method. Utility of results, adherence to logical principles, and the danger of overformalization are among the limitations inherent in the method.

3. *Recognition and constructions of conic sections without the use of the discriminant*, by Albert Furman, Kansas State College.

The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

can be written in the form

$$y = -\frac{hx + f \pm A}{b}$$

where

$$A = (h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc.$$

The graph of (1) can thus be constructed as the composite $y = y_1 + y_2$, where $y_1 = -(hx+f)/b$ and $y_2 = \pm A$. The latter equation represents the conic section

$$(h^2 - ab)x^2 - b^2y^2 + 2(hf - bg)x + (f^2 - bc) = 0. \tag{2}$$

Classification may be effected by use of the principle that equations (1) and (2) agree as to type and degeneracy or non-degeneracy.

4. *Vector methods in analytic geometry*, by L. M. Reagan, University of Wichita.

It was pointed out that vector methods, widely used with great effectiveness in Mechanics and other branches of applied mathematics are equally effective in the development of many branches of pure mathematics, and there is a growing tendency to use them, especially in Trigonometry and Geometry, including Analytic Geometry. Some texts for use in standard courses in Analytic Geometry and Calculus, based on vector methods, have appeared recently, and while not widely adopted as yet, have a great deal of merit. The simplicity of vector methods in Analytic Geometry was illustrated with several examples. In particular, the development of the standard, intercept, slope and y-intercept forms of the straight line, condition for collinearity of three points, condition that four points lie in a plane, equation of the circle in various positions, and equation of a tangent to a circle, were treated by use of vectors.

5. *A characterization of a sphere*, by Robert Schatten, University of Kansas.

Point-sets whose all plane sections are Jordan curves are considered in the three-dimensional euclidean space. In particular, a point-set whose every plane section is the circumference of a circle characterizes the surface of a sphere.

ANNA MARM, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at The Rice Institute, April 23–24, 1948. Professor R. S. Underwood, Chairman of the Section, presided.

The following members of the Association were present: Ina Mae Bramblett, H. E. Bray, H. D. Brunk, J. W. Calkin, J. A. Daum, Alice C. Dean, E. H. Hanson, H. M. Hardy, E. R. Heineman, N. C. Hunsaker, H. A. Luther, Hazel Mason, Harlan C. Miller, Edith Morgan, Albert Newhouse, J. W. Querry, C. B.

Rader, L. W. Ramsey, Dorothy Rees, C. R. Sherer, M. M. Slotnick, F. W. Sparks, E. J. Stulken, Jennie Tate, H. E. Taylor, W. B. Temple, F. E. Ulrich, R. S. Underwood, and T. J. White.

The following officers were elected for next year: Chairman, F. E. Ulrich, The Rice Institute; Vice Chairman, H. J. Ettlinger, University of Texas; Secretary-Treasurer, C. R. Sherer, Texas Christian University. The next meeting will be held at Denton, Texas under the joint sponsorship of North Texas State Teachers College and Texas State College for Women.

An informal dinner was held at The Rice Institute, April 23 at 7 P.M. Dr. H. A. Wilson, Professor Emeritus of Physics, The Rice Institute, gave a very interesting address.

The following papers were presented:

1. *The application of group theory to the normal vibration of a cubic crystal*, by Dr. W. V. Houston, President, The Rice Institute.

This paper was delivered by special invitation.

2. *On a differential equation of infinite order*, by Dr. H. D. Brunk, The Rice Institute.

3. *College level problems from seismic prospecting*, by Dr. E. J. Stulken, Geophysical Service, Inc.

4. *Some applications of extended analytic geometry*, by Dr. R. S. Underwood, Texas Technological College.

5. *The exact square root of any positive number by construction*, by P. J. C. Lauback, Westminister College, introduced by the Secretary.

6. *Freshman mathematics for business students*, by C. R. Sherer, Texas Christian University.

7. *The training of mathematics teachers for secondary schools*, by Dr. E. H. Hanson, North Texas State Teachers College.

C. R. SHERER, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held at the University of Kansas City, Kansas City, Missouri, on April 23, 1948. Professor P. R. Rider presided.

About sixty persons were in attendance, including the following twenty-eight members of the Association: L. M. Blumenthal, C. H. Brown, C. B. Burcham, L. H. Cutting, C. E. Denny, W. C. Doyle, D. H. Erkiletian, Jr., G. M. Ewing, D. G. Ewy, C. A. Johnson, L. O. Jones, J. C. Koken, Sister M. Pachomia Lackay, L. E. Laird, Walter Leighton, C. W. Mathews, E. F. Moore, S. T. Parker, A. D. Pierson, G. B. Price, R. M. Rankin, P. R. Rider, J. S. Rosen, R. G. Sanger, G. W. Smith, R. G. Smith, C. B. Tucker, G. B. Van Schaack.

The University of Kansas City was host at a luncheon, at which Dean Norman Royall welcomed those present.

The following officers were elected for the coming year: Chairman, P. R.

Rider, Washington University; Vice-Chairman, W. C. Doyle, Rockhurst College; Secretary, C. W. Mathews, Washington University.

The program was arranged by Professor J. S. Rosen of the University of Kansas City. The following papers were presented:

1. *Almost periodic functions*, by Mr. Asger Aaboe, Washington University, introduced by Professor Walter Leighton.

The theory of almost periodic functions of a real variable, created by Harold Bohr, is briefly outlined.

2. *Convergence regions for continued fractions*, by Professor W. J. Thron, Washington University, introduced by Dr. C. W. Mathews.

The historical development of the subject is outlined. This includes the contributions of Worpitzky, A. Pringsheim, H. B. Van Vleck, H. S. Wall, W. Leighton, and the author. The second part of the paper consists of a brief discussion of the principal methods of proof used.

3. *The definition of the Dirac δ-function*, by Professor G. M. Ewing, University of Missouri.

The so-called Dirac δ-function not only appears in the literature as far back as Kirchhoff, but is introduced in many recent books, usually with an apology, followed by the statement that nevertheless valid results are obtained by following the stated conventions. The present paper introduces a certain class K of functions $\delta_\epsilon(t)$, and points out that the limits of a number of integrals involving δ_ϵ provide interpretations for familiar statements about the δ-function. Such results are independent of the choice of δ_ϵ in K. Linear differential equations $F(D)x=\delta_\epsilon(t)$ are also considered.

4. *The exponential function in applied science*, by Professor Herman Betz, University of Missouri, introduced by Professor L. M. Blumenthal.

The assumptions usually made in describing certain theorems occurring in the natural sciences by means of the exponential function can be considerably relaxed. This paper indicates how this may be done, even in undergraduate courses.

5. *Rational right triangles*, by S. G. Campbell, University of Kansas City, introduced by Professor J. S. Rosen.

The speaker discussed the problem of finding a method of obtaining a series of primitive rational right triangles, the ratios of whose sides are increasingly close approximations to the ratios of the sides of any given irrational right triangle. The method used involves use of a theorem by Euler for expressing the roots of a quadratic equation in terms of a continued fraction. He also dealt with the problem of finding a general method of locating the series of all primitive rational right triangles with a given relation between the sides (as a given difference between any two sides.) The method developed uses a series (recurrent) with scale which is also satisfied by consecutive coefficients generated by a particular rational fraction.

6. *A theorem on determinants*, by Professor G. B. Price, University of Kansas.

This paper contains a simple proof of the following theorem and some of its generalizations: If (a_{ik}), $i, k=1, 2, \cdots, n$, is any matrix of complex numbers such that

$$|a_{ii}| > \sum_{k \neq i} |a_{ik}|, \qquad i = 1, 2, \cdots, n$$

then the determinant $|a_{ik}| \neq 0$. This theorem has been rediscovered repeatedly ever since Levy

published the first proof of it in 1881—two first discoveries are reviewed in the 1947 volume of *Mathematical Reviews*. The theorem is one of considerable importance in both pure and applied mathematics, and its history emphasizes the need for a better dissemination of known results.

7. *The importance of computational technique in applied mathematics*, by Y. L. Luke, Midwest Research Institute, Kansas City, Missouri, introduced by Professor J. S. Rosen.

The complexity of many problems in applied mathematics makes important the need for computational technique. Some examples met in actual practice are given. The need for courses in applied mathematics in the undergradutate school is emphasized.

8. *A new trigonometric shifting theorem*, by Professor Eugene Stephens, Washington University, introduced by Professor P. R. Rider.

We may shift $\sin ax$, $\cos ax$, $\sinh ax$, $\cosh ax$ across a linear differential operator of the form $F(D)=\sum a_n D^n$ in the following manner:

(A) $F(D)\cdot\sin ax \equiv \sin ax\cdot F(D+aC), \qquad C \equiv \cot ax$

(B) $F(D)\cdot\cos ax \equiv \cos ax\cdot F(D-aT), \qquad T \equiv \tan ax$

(C) $F(D)\cdot\sinh ax \equiv \sinh ax\cdot F(D+aC_h), \qquad C_h \equiv \coth ax$

(D) $F(D)\cdot\cosh ax \equiv \cosh ax\cdot F(D+aT_h), \qquad T_h \equiv \tanh ax$

If in the right hand side of these operator identities there appears anywhere C^2, T^2, C_h^2, T_h^2, each can be replaced by -1. These are considered as constants and as commutative with the operator D. The resulting operations on a given subject are simplified by the elementary operational theorems. In the final results the C, T, C_h, T_h are replaced by their respective trigonometric forms.

9. *The real representation of imaginary loci*, by L. E. Laird, Kansas State Teachers College.

This paper presents a method of representation of the pairs of complex numbers (complex points) which satisfy an equation $w=f(z)$, where $w=u+iv$ and $z=x+iy$. The method sets up a one-to-one correspondence between the complex points and real lines in space by use of Plucker's line coordinates. If a functional relationship is assumed between x and y, a single infinity of real lines is determined which constitute a ruled surface.

10. *The convergence in probabilities of statistical sequences*, by Dr. Maria Castellani, University of Kansas City, introduced by Professor J. S. Rosen.

The theory of fitting a curve to a given statistical sequence of data may be considered as a special type of convergence in probabilities. The purpose of the paper is to show how the several definitions of convergence may suit the statistical data, and how results obtained may be tested. The continuity in probabilities of statistical sequences is also considered with special regard to the case in which the sequence converges to a parabola or to a sinusoid.

11. *Mathematics placement tests at the University of Kansas*, by Professor G. W. Smith, University of Kansas.

Professor Smith points out the need for some kind of a placement test in mathematics, and explains the plan used at the University of Kansas. He points out that in September 1946 about twenty per cent of the entering freshmen were affected by the test. The classes were made more nearly uniform, and the college algebra course was strengthened.

12. *Preparation for college mathematics*, by Professor W. C. Doyle, Rockhurst College.

Professor Doyle conducted a panel discussion on ways and means to encourage high schools to strengthen their mathematics programs. It was remarked that colleges should better coordinate what they expect of freshmen, and make better use of placement tests.

P. R. RIDER, *Secretary*

APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-first annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado A. and M. College, Fort Collins, Colorado, April 23 and 24, 1948. Professor H. T. Guard presided at all the sessions.

Among the eighty-one persons who registered were the following thirty members of the Association: C. F. Barr, D. L. Barrick, W. G. Brady, J. R. Britton, F. M. Carpenter, A. G. Clark, G. S. Cook, A. B. Farnell, F. N. Fisch, H. T. Guard, Leota C. Hayward, I. L. Hebel, Ruth I. Hoffman, LeRoy Holubar, Burrowes Hunt, J. A. Hurry, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, M. L. Madison, A. E. Mallory, W. K. Nelson, Greta Neubauer, K. L. Noble, Nathan Schwid, S. R. Smith, L. C. Snively, V. J. Varineau, Lillie C. Walters.

At the business meeting, the officers elected for the coming year were: Chairman, Professor I. L. Hebel, Colorado School of Mines; Vice-Chairman, Professor A. J. Lewis, University of Denver; Secretary-Treasurer, Professor J. R. Britton, University of Colorado. Professor A. J. Lewis was also elected Sectional Governor for a term of three years. A resolution commending Professor Abraham Wald for the excellence of his invited addresses was unanimously adopted.

The program of papers presented was as follows:

1. *A method of defining the real number system*, by Robert Howerton, University of Denver, introduced by A. J. Lewis.

2. *A slow-motion algorithm*, by Burrowes Hunt, University of Colorado.

The euclidean algorith for two relatively prime integers $a>b$ which leads to the equations

$$a = q_1 b + r_1, \qquad b = q_2 r_1 + r_2, \cdots$$

is modified by taking each $q_i = 1$. This algorithm terminates if and only if a and b are successive integers of the Fibronacci sequence. The least positive remainder is 1 if and only if, as a regular continued fraction, $a/b = (1; 1, \cdots, 1, k)$, k being an arbitrary positive integer. If $a/b = (1; 1, \cdots, 1_n, a_0, a_1, \cdots, a_k)$, and $(a_0; a_1, \cdots, a_k) = q/r$, the algorithm gives a least positive remainder r on the nth step.

3. *A note on expansion of determinants*, by Professor W. R. Eikelberger, University of Denver, introduced by A. J. Lewis.

4. *An approximation to the solution of a non-linear partial differential equation*, by Professor Nathan Schwid, University of Wyoming.

The differential equation of heat conduction

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right)$$

is non-linear when the conductivity K is a function of u. Here the diffusivity $K/c\rho$ is taken as $\alpha^2+\beta^2 u$, where the ratio β^2/α^2 is small. The heat is considered as flowing in the x direction only in a plate of finite width. The solution of the resulting non-linear equation is approximated by a modification of a method given by Kirchoff about 1890 in the *Annalen der Physik* for an analogous problem involving flow of heat in one direction in a semi-infinite solid with the same type of diffusivity as above.

5. *Expansion of functions in combinations of generalized hypergeometric functions*, by Professor Leonard Bristow, University of Wyoming, introduced by C. F. Barr.

The purpose of this paper is to obtain the expansion of a suitable arbitrary function of a real variable in a series of solutions of a self adjoint differential equation of the Cauchy or Euler type containing a parameter. There are one-point boundary conditions (taken to be at $x=1$) together with regularizing conditions at the regular singular point (taken to be at $x=0$) of the differential equation. A Green's function is obtained. Fourier series, Fourier-Bessel, and Dini expansions in Bessel functions are obtained as special cases.

6. *Introduction to sequential analysis*, by Professor Abraham Wald, Columbia University.

7. *Principles of sequential analysis*, by Professor Abraham Wald, Columbia University.

These papers by Professor Wald were invited addresses.

8. *Ivory Towers*, by Miss Ruth I. Hoffman, Denver, Colorado.

There is need for college mathematics teachers to step down and become acquainted with the content of secondary mathematics, the problems and factors that influence the type of courses offered, and the quality of these courses. The university people should know of the valiant struggle that mathematics teachers in secondary education are making to keep up with modern educational trends, and to meet the needs of the present student body while still teaching sound mathematics, and even showing the beauty, as well as the usefulness, of mathematics.

9. *On certain equations involving radicals*, by Harlan Bartram, University of Colorado, introduced by A. J. Kempner.

The following problem was discussed: Given

$$\sqrt{a+bi}+\sqrt{c+di}=f, \qquad i=\sqrt{-1}.$$

If a, b, c, and d are real, and the signs of the radicals are properly chosen, when will f be real? The necessary and sufficient condition for this was found to be that $(b^2-d^2)^2=4(a-c)(ad^2-cb^2)$.

10 *Some teaching devices in undergraduate mathematics*, by Professor S. R. Smith, University of Wyoming.

Experience has shown that the majority of students entering college have difficulty in mathematics courses. At least part of this difficulty is due to lack of organization of their work, particularly in the solution of problems, and to the interpretation of the solutions found. Teaching devices, not necessarily new, are suggested to facilitate the solution of systems of quadratic equations, the discussion and sketching of plane curves in analytic geometry, and the application of the first and second derivatives in calculus.

11. *The book says so*, by Professor A. E. Mallory, Colorado State College of Education.

In this paper attention is called to the importance of teaching the connection between the solution of equations and the concept of a function.

12. *What constitutes good mathematics for undergraduates?* by Professor A. G. Clark, Colorado A. and M. College.

The author used the article, *Can We Teach Good Mathematics to Undergraduates?* by R. G. Helsel and T. Radó, which appeared in the January, 1948 issue of this MONTHLY, as the basis for his discussion. He agreed in part with the opinions of Helsel and Radó, but the extent of the agreement was dependent upon the connotation given the word "elegant," a term which mathematicians seem to have appropriated. The concept of "efficient" mathematics for undergraduates was presented.

13. *Report on entrance requirements*, by Professor A. J. Kempner, University of Colorado.

This paper was a brief report on the discussions which were held and the resolutions which were passed at the meetings of the Mathematical Association at Athens, Georgia, and the National Council of Teachers of Mathematics at Indianapolis, Indiana, in connection with the problems of lowering college entrance requirements and standards in general, and those pertaining to mathematics in particular.

Following a short discussion of the last paper, the following resolution was unanimously adopted: The Rocky Mountain Section of the Mathematical Association of America approves whole-heartedly the recent action of the Mathematical Association and the National Council of Teachers of Mathematics in expressing their desire for the closest cooperation in the critical problems confronting secondary and college mathematics.

J. R. BRITTON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

ALLEGHENY MOUNTAIN West Virginia University, Morgantown, May 7, 1949

ILLINOIS, Bradley University, Peoria, May 13–14, 1949

INDIANA, University of Notre Dame, Spring, 1949

IOWA, Drake University, Des Moines, April 15–16, 1949

KANSAS, Manhattan, April 2, 1949

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LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK Brooklyn College, April 9, 1949

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NEBRASKA, Lincoln, May, 1949

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 2, 1949

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, March 25–26, 1949

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949

SOUTHEASTERN, University of Alabama, University, March 18–19, 1949

SOUTHERN CALIFORNIA, John Muir Junior College, Pasadena, March 12, 1949

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UPPER NEW YORK STATE, University of Buffalo, April 30, 1949

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PREFACE

This *Outline* originally consisted of two Lectures delivered at the University of Minnesota in September 1931, at a Summer School for Engineering Teachers organized by The Society for the Promotion of Engineering Education. In February 1932 the Society published an edition of 1030 copies of the *Outline* as number 18 of its *Selected Papers*. By the end of 1933 this edition was exhausted, and since it was contrary to the policy of the Society to publish second editions of such *Papers*, all rights in the publication were made over to the Mathematical Association of America, Inc., which felt that it might be rendering a service to teachers of the History of Mathematics, and to others, by publishing a second revised and enlarged edition of the *Outline*. This appeared in July 1934.

The similarly revised and enlarged third, fourth and fifth editions appeared successively in June 1936, February 1939, and June 1941. And now with the fifth edition entirely exhausted The Mathematical Association is about to publish the largest edition yet, 7500 copies, and to send it forth as one of the *Papers* in memory of Professor Slaught—one of my most highly prized friends for the last twenty years of his life. No longer hampered by changes in standing type I have devoted my best efforts to making this Memorial worthy. Every statement in the main text was carefully reconsidered, every reference in the Literature List and Notes (LLN) was once more consulted; as a result many errors were eliminated. Moreover it has been my endeavor to present the very latest results of scholarly research, and in the 328 LLN greatly to augment the list of topics discussed, and sources where material may be found for notably amplifying the text. This material ought to be of special value to the large group of teachers of the History of Mathematics in this country. According to a survey in *Scripta Mathematica*, v. 2, pp. 161–165, 1934, courses in this field were offered during the years 1932–34 in 161 institutions: Liberal Arts Colleges and Universities (125), Teachers' Colleges and Normal Schools (32), and Technical Schools (4).

The main text of this edition has undergone very many changes but it has been only slightly enlarged, through such things as the increase of guiding material connected with Babylonian and Arabic mathematics, the transfer from LLN to the text of remarks on achievements of PLATO, PACIOLI, STEVIN, KEPLER, and SACCHERI, and through the addition, among others, of notes about THEON, PROCLUS, and JAMES GREGORY. That references to astronomy are desirably more numerous than in earlier editions will be observed.

There are a good many passages where current histories are amplified, or changed, or corrected. Perhaps the most striking one is the placing of HERON, MENELAUS, and DIOPHANTUS together in the first century of our era. On most questions, and especially those which are debatable, authorities are always noted. It is not claimed that DIOPHANTUS certainly flourished in the first century, but merely that evidence definitely points to this as a possibility, while denying the correctness of arguments assigning him to the third century.

Should the scholar happen to turn these pages he should understand that in many LLN excessive details and numerous odd bits of information were not assembled for him, but intended to help the teacher in elaborating and vivifying his material for class use. The Index of Names has been improved by the inclusion of many dates.

In preparing this edition I have had the privilege of constant consultation with my colleague and friend Professor OTTO E. NEUGEBAUER, the greatest living authority on ancient mathematics and astronomy. My debt to him in this connection is indeed very great. Here, as in earlier editions and in other publications, I am also deeply indebted to my friend Mr. S. A. JOFFE, who added materially to a desirable final form of the printer's manuscript and proof by very numerous suggestions for improvement.

I should be grateful to any one informing me of inaccuracies found in this *Outline*.

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INTRODUCTORY NOTE

An attempt is here made to give indications of the development of mathematics before the nineteenth century, and to refer briefly to some developments of the nineteenth and twentieth centuries in connection with topics usually discussed at undergraduate colleges. Chinese and Japanese mathematics are not considered in such a skeleton survey, and the reference to mathematics of the Hindus is brief.

It was felt that consideration of only what was most important from a purely mathematical point of view would not be desirable in the survey, partly because many of the most important things cannot, in a few sentences, be made intelligible to one wholly unfamiliar with the subject, and partly because the quotation of complicated results in a lecture is not possible. Hence many minor items in the development of the science have been introduced in order that interest may be sustained through knowing the historical setting for certain simply stated, or well-known, results.

While an endeavor has been made to present the results of the latest investigations, no claim for originality can be made; phrases, quotations, and adaptations from many sources have been used constantly. In the Literature List and Notes all such sources are indicated and in the text many references are given to them in order that the reader may if he choose inform himself more fully concerning matters discussed. We indicate the chief worthwhile sources in English (many of them popular in style); but a scholarly grasp of any field would, however, necessitate the consultation of literature in other languages, and many sources of this kind have been listed.

SYNOPSIS

Following SMITH (see Literature List and Notes [4]=LLN 4) I have used such an abbreviation as −1850 for 1850 B.C. in the synopsis which follows, and have associated with the name of each mathematician a single definite date indicating a year about when he flourished or when he published some outstanding work.

I. History of Mathematics before 1600 A. D.

A. Babylonian and Egyptian Mathematics and Astronomy 3100 B.C. to 1 B.C.

Babylonian: Geometry and algebra, solution of quadratic and cubic equations, tables, compound interest, square root approximations, series texts and simultaneous equations, astronomy, NABURIANU and KIDINNU, −2000 to −1

Egyptian: Great pyramid, −2900; obelisks; timepieces; Moscow papyrus −1850; RHIND papyrus −1650; astronomy

B. Greek Mathematics 600 B.C. to 600 A.D.

THALES (−600): Geometry

PYTHAGOREANS (−540 to −400): Geometry, arithmetic, music

Famous problems (−450):

(*a*) Trisection of an angle: HIPPIAS of Elis (−425) with quadratrix, NICOMEDES (−240) with conchoid

(*b*) Squaring the circle: DINOSTRATUS (−350) and NICOMEDES (−240) with quadratrix, ARCHIMEDES (−225) with his spiral

(*c*) Duplication of the cube: HIPPOCRATES of Chios (−460) with two mean proportionals, ARCHYTAS (−400) with surfaces, DIOCLES (−180) with cissoid

ZENO of ELEA (−450): paradoxes

PLATO (−380): mathematics important in training mind

THEAETETUS (−375): Incommensurables, regular solids

EUDOXUS (−370): Incommensurables, method of exhaustion, astronomy

MENAECHMUS (−350): Conic sections

EUDEMUS (−335): historian of science

AUTOLYCUS of Pitane (−330): Spherical geometry [75]

EUCLID (−300): Elements, perfect numbers, conics, optics

ARISTARCHUS of Samos (−287): Copernican system

ARCHIMEDES (−240): Geometry, arithmetic, mechanics, hydrostatics

ERATOSTHENES (−230): Size of earth, sieve

APOLLONIUS of Perga (−225): Conic sections, plane loci, tangencies, epicycles, eccentrics

HIPPARCHUS (−140): Trigonometry, solar and lunar movement, star catalogue

THEODOSIUS of Bithynia (between −180 and −25): Spherical geometry [75]

HERON of Alexandria (+75): Pneumatica, cube root, torus volume

DIOPHANTUS of Alexandria (perhaps 75): Diophantine equations

MENELAUS of Alexandria and Rome (100): Spherical geometry and trigonometry

PTOLEMY of Alexandria (150): Trigonometry, projections, lunar and planetary theory, star catalogue, geography, music, optics

PAPPUS of Alexandria (300): Mathematical Collection, Commentaries on Euclid's Elements, bk. 10, and Ptolemy's *Almagest*

THEON of Alexandria (390): Edition of Euclid's Elements, commentary on the *Almagest* assisted by his daughter Hypatia

PROCLUS (470): Valuable commentary on Euclid's Elements and his Hypotyposis of Astronomical Hypotheses

Hindu Mathematics before 600: [113]

C. Hindu, Arabic, Persian Mathematics 600 to 1200

BRAHMAGUPTA (630): Area of an inscribed quadrilateral

BHĀSKARA (1150): Linear and quadratic indeterminate equations

Hindu-Arabic numerals (−300 to +876)

AL-KHOWÂRIZMÎ (820): Algebra, numeration, algorism, tables, astronomy

ABÛ'L WEFÂ (980): Geometric constructions with ruler, and compasses having a fixed opening, trigonometry, tables

OMAR KHAYYÀM (1100): Solutions of cubic equations, calendar

D. European Mathematics 1200 to 1600

LEONARDO of Pisa (Fibonacci) (1202): *Liber Abaci*, *Liber Quadratorum*, *Practica Geometriae*, *Flos*

NASÎR ED-DÎN AL-TÛ'SÎ (1250): First independent work on trigonometry, commentary Euclid's Elements

ULUGH BEG (1435): Trigonometric tables

REGIOMONTANUS (JOHANN MÜLLER) (1470): Trigonometry, astronomy

First printed arithmetic (1478)

LUCA PACIOLI (1494): Sūma (including double-entry bookkeeping), "divine proportion"

SCIPIONE DEL FERRO and ANTONIO FIOR (1506): Cubic equation

TARTAGLIA (NICOLO of Brescia) (1535): Cubic equation

GIROLAMO CARDANO (1545): Algebra

FERRARI (1545): Biquadratic equation

RHETICUS (1550): Tables of natural trigonometric functions
NICOLAUS COPERNICUS (1530): System of the universe
SIMON STEVIN (1590): Systematic explanation decimal fractions, compound interest table, statics, hydrostatics
ROBERT RECORDE (1542): Arithmetic, geometry, algebra [162]
VIETA (1580): Algebra, trigonometry, notation
First mathematical book published in the New World (1556)

II. History of Mathematics after 1600

A. The Seventeenth Century

JOHN NAPIER (1614): Logarithms, trigonometry, computing rods
HENRY BRIGGS (1624): Logarithms to the base 10
JOHANN KEPLER (1610): Laws for motion of planets, volumes, maxima and minima, star polyhedra, principle of continuity in geometry, logarithms
GALILEO GALILEI (1600): Falling bodies, projectiles, cycloid, momentum
THOMAS HARRIOT (1600): Algebra, symbolism
WILLIAM OUGHTRED (1630): *Clavis Mathematicae*, algebra, slide rule, notations, first table of natural logarithms
GIRARD DESARGUES (1640): Projective geometry, treatise on singing
ÉTIENNE PASCAL (1635): Limaçon
BLAISE PASCAL (1650): Conics, computing machine, cycloid, theory of probabilities
GEORGE MOHR (1672): Constructions with compasses only
RENÉ DESCARTES (1637): Analytic geometry, equations, $F+V=E+2$, folium, ovals
PIERRE DE FERMAT (1635): Theory of numbers, maxima and minima, theory of probabilities, analytic geometry
BONAVENTURA CAVALIERI (1635): Indivisibles
JAMES GREGORY (1670): Binomial theorem, "Taylor's" theorem, expansion of functions into series, integrations
JOHN MACHIN (1706): π in terms of inverse tangents [201]
CHRISTIAAN HUYGENS (1670): Circle quadrature, theory of probabilities, evolutes, pendulum clocks, catenary
ISAAC BARROW (1670): Editor of works of EUCLID, APOLLONIUS, THEODOSIUS; geometry, differentiation, optics [210]
JOHN WALLIS (1650): Algebra, imaginary numbers, length of curves, π as infinite product
ISAAC NEWTON (1680): Fluxions, dynamics, hydrostatics, hydrodynamics, gravitation, cubic curves, series, numerical equations, imaginary roots, challenges
GOTTFRIED WILHELM LEIBNIZ (1682): Newton controversy, calculus, determinants, polynomial expansions, notations
JAMES I BERNOULLI (1690): Isochronous curves, Euler's spiral, logarithmic spiral, calculus of probability
EDMOND HALLEY (1690): [211]; GILLES PERSONE DE ROBERVAL (1640): [197]; EVANGELISTA TORRICELLI (1640): [197]

B. The Eighteenth Century

ANTOINE PARENT (1700): ALEXIS CLAIRAUT (1731), and JAKOB HERMANN (1732–33): Analytic geometry of three dimensions
GIROLAMO SACCHERI (1733): Noneuclidean geometry prelude
ABRAHAM DEMOIVRE (1720): Actuarial mathematics, theory of probabilities, trigonometry with complex quantities, $m! \approx (2\pi m)^{\frac{1}{2}} e^{-m} m^m$
BROOK TAYLOR (1720): Expansion in series, finite differences, vibrating string
COLIN MACLAURIN (1740): Fluxions, organic description of curves
LEONARD EULER (1750): Publications, notations, $e^{i\pi}+1=0$, Euler's line, biquadratic

equation, indicatrix, vibrating string, calculus of variations, Euler's spiral, beta and gamma functions, Euler's constant

CASPAR WESSEL (1797): Imaginary quantities; JEAN CHARLES BORDA (1799); JEAN CHARLES CALLET (1795), and JOSEPH JERÔME FRANÇOIS DE LA LANDE (1775): Mathematical tables; JEAN ÉTIENNE MONTUCLA (1775): Histories of mathematics [246]

JOSEPH LOUIS LAGRANGE (1780): Calculus of variations, differential equations, mechanics, calculus foundations, theory of numbers, numerical equations

JOHANN HEINRICH LAMBERT (1770): *Freye Perspective*, parabola property, π, e^x, tan x, irrational, hyperbolic functions, map projection

WILLIAM WALLACE (1798): Geometry [251]

GASPARD MONGE (1795): Descriptive geometry, differential geometry

C. The Nineteenth Century and Later

PIERRE SIMON LAPLACE (1805): Astronomy, celestial mechanics, probabilities, differential equations

ADRIEN MARIE LEGENDRE (1805): Theory of numbers, elliptic functions and integrals, exercises in integral calculus, geometry

JEAN ROBERT ARGAND (1806): Imaginary quantities [246]

JOSEPH FOURIER (1812–1822): Series expansions, theory of equations, physics [246]

JEAN BAPTISTE DELAMBRE (1820): Histories of astronomy, trigonometry [246]

Noneuclidean geometry: BOLYAI and LOBACHEVSKY (1825)

KARL FRIEDRICH GAUSS (1820): Polygon constructions, theory of numbers, differential geometry, algebraic equations, etc.

Calculating prodigies: ZACHARIAS DASE (1855)

Projective and modern geometry: VICTOR PONCELET (1830) and JAKOB STEINER (1840); duality, CHARLES J. BRIANCHON (1806)

Trisection of an angle; duplication of cube proved impossible (1837)

Inversion: STEINER (1824), QUETELET (1825), linkages

Algebraic solution of the general quintic equation impossible: NIELS HENRIK ABEL (1824)

Approximating and locating roots of numerical equations: NEWTON (1669); RUFFINI (1804), HORNER (1819), FOURIER (1820), STURM (1829), CAUCHY (1831). See also [246, 285]

Convergence of series: NEWTON, GAUSS, CAUCHY

JOSEPH LIOUVILLE (1850): Transcendental numbers [253]

CHARLES HERMITE (1882): Transcendence of e [254]

Geometry of the triangle: LEMOINE and BROCARD (1873–75, and later)

Geometrography: LEMOINE (1888 f.)

Vector analysis: GIBBS (1881–84), HEAVISIDE (1891)

Quaternions: W. R. HAMILTON (1853), Ausdehnungslehre: H. G. GRASSMANN (1844, 1862) [292]

Determinants: LEIBNIZ, LAGRANGE, CAUCHY, JACOBI

Mathematical Tables: U. S. Nat. Res. Council Comm. on Math. Tables and Other Aids to Computation; U. S. Nat. Bureau of Standards Comm. on Math. Tables; Br. Assoc. Adv. Sci. [now Royal Soc.] Comm. Math. Tables, J. W. L. GLAISHER and CAYLEY (1873–1883, 1911); HENDERSON (1926), THOMPSON (1924–1948), ANDOYER (1915–1918), LEHMER (1909, 1914), DE MORGAN (1842–1861), DUFFIELD (1896), VEGA (1794). Also [293–302a]

Surveys of Pure and Applied Higher Mathematics: E. W. BROWN (1923), F. KLEIN (1926–27), R. COURANT (1926), R. S. WOODWARD (1900), BELL (1945), STRUIK (1948), Amer. Math. Soc. Semicentennial Addresses and History (1938), SMITH & GINSBURG (1934), CAJORI (1890), BEATTY (1939), PRASAD, BELL (1937), COOLIDGE (1940), KÖTTER (1901), LORIA (1931), *Teubners Sammlung, Grundlehren, Ergebnisse, Mémorials, Cambridge Tracts*. ISAAC BARROW. See also [303–312]

HISTORY OF MATHEMATICS BEFORE THE SEVENTEENTH CENTURY

A. BABYLONIAN AND EGYPTIAN MATHEMATICS AND ASTRONOMY 3100 B.C. TO 1 B.C. [13–26]

The period covered by this first lecture on the history of mathematics will be about 4700 years; for we shall begin by noting that in one of the great museums at Oxford is a royal mace [36] of 3100 B.C. on which there is a record of 120,000 prisoners, of 400,000 captive oxen, and of 1,422,000 captive goats. These numbers, written in Egyptian hieroglyphs, show that already in this ancient time not only was the decimal system of numeration, but also a method for writing very large numbers, thoroughly established. It is interesting to speculate on how many thousand years earlier must have been *the beginnings* when great communities, of a millennium later, with a highly developed social order, calling for the frequent use of mathematics of taxation, barter, interest, and of large numbers, were replaced by small groups of primitive people for whose simple needs such number words as "one," "two," and "heap" would wholly suffice [4, 19].

Any general primitive notions of this kind probably long antedated the execution of the recently discovered finest art-work of primitive man, reputed to be from 25,000 to 50,000 years old [18]. It is a sculpture of a white rhinoceros with a swarm of attendant tick-birds, hammered into a slab of basaltic rock, and showing extraordinary power of form, line, and perspective—elements intimately allied to things mathematical.

But there were other people besides the Egyptians who contributed notably to early mathematics. I refer to the non-Semitic Sumerians who lived just north of the Persian Gulf and south of the Semitic Akkadians, and who for many centuries prior to 2500 B.C. were generally predominant in Babylonia, but were absorbed into a larger political group by about 2000 B.C. One of the greatest of the Sumerian inventions was the adoption of cuneiform script; notable engineering works of the Babylonians, by means of which marshes were drained and the overflow of rivers regulated by canals, went back to Sumerian times, as also a considerable part of their religion, their law, and their mathematical notation [13, 17, 19].

An extraordinary number of tablets show that the Sumerian merchant of about 2000 B.C. was familiar with such things as weights and measures, bills, receipts, notes, and accounts. Long before coins were in use (7th century B.C.) it was common custom to pay interest for the loans of produce, or of a certain weight of a precious metal. Tablets indicate that the rate of interest varied from 20 to 30 per cent, the higher rate being charged for produce.

Sumerian or Babylonian arithmetic was essentially sexagesimal, although there are influences of a decimal system. Hence the general numeral might be written in the form $a_n \cdot 60^n + a_{n-1} \cdot 60^{n-1} + \cdots + a_1 \cdot 60 + a_0 \cdot 60^0 + b_{-1} \cdot 60^{-1} + \cdots + b_{-m} \cdot 60^{-m}$ or $a_n, a_{n-1}, \cdots a_1, a_0, b_{-1}, \cdots b_{-m}$. In particular 31, 6, 15 might

equal 111975. The symbol for unity was also the symbol for any one of the numbers 60^n where n is a positive or negative integer. Thus 31, 6, 15 might mean not only 111975, but also 6718500 or 1866 1/4 or 31 1/10 1/240, according as the 15 is taken as multiplied by 1, 60, or 1/60, or 1/3600. The uncertainty in this regard introduces certain difficulties in interpreting Babylonian mathematical texts. Another uncertainty was introduced through the fact that where a zero enters may be determined only by the context; so that, for example, 11, 7 or 11, , 7 might stand for 39607. But a blank space does not always mean zero in Old Babylonian tablets. So far as is at present known, no special symbol for zero was used by the Babylonians before about 400 B.C., and this symbol was used to indicate a zero both in the interior and at the end of the number.

In the field of geometry the Babylonian of 2000 to 1600 B.C. used the following results in concrete cases, from which we have to infer that they were familiar with the general rules:

1. The area of a rectangle is the product of the lengths of two adjacent sides.
2. The area of a right triangle is equal to one-half the product of the lengths of the sides about the right angle.
3. The sides about corresponding angles of two similar right triangles are proportional.
4. The area of a trapezoid with one side perpendicular to the parallel sides is one-half the product of the length of this perpendicular and the sum of the lengths of the parallel sides.
5. The perpendicular from the vertex of an isosceles triangle on the base bisects the base. The area of the triangle is the product of the lengths of the altitude and half the base. Indeed the Babylonians may have thought of this result for the area of a triangle other than right or isosceles, since such a triangle may be regarded as made up of adjacent or overlapping right triangles; but there is no known example of this use of the formula.
6. The "Pythagorean" theorem; for example, for triangles with sides corresponding to the numbers 3, 4, 5; 5, 12, 13; 8, 15, 17; 20, 21, 29; and many more; see after no. 11, below.
7. The angle in a semi-circle is a right angle [55].
8. The length of the diameter of a circle is one-third of its circumference ($\pi = 3$). The area of a circle is 1/12 of the square of its circumference (correct for $\pi = 3$).
9. The volume of a rectangular parallelopiped is the product of the lengths of its three dimensions, and the volume of a right prism with a trapezoidal base is equal to the area of the base multiplied by the altitude of the prism.
10. The volume of a right circular cylinder is the area of its base multiplied by its altitude.
11. The volume of the frustum of a cone, or of a square pyramid, is equal to its altitude multiplied by one-half the sum of the areas of its bases. It has been conjectured that the Babylonians had also the equivalent of an exact

formula for the volume in the case of a square pyramid, namely

$$V = h\left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right],$$

where a and b are the lengths of the sides of the square bases [20]. This was known to HERON of Alexandria 1700 years later, and reduces to the extraordinary formula apparently known to the Egyptians.

The most remarkable recent discovery in connection with Babylonian mathematics was that made by Professor NEUGEBAUER, in tablet PLIMPTON 322, dated -1900 to -1600, at Columbia University, which contains a table of "Pythagorean" numbers [16]. Let l denote the longer, s the shorter side of a right triangle, and h its hypotenuse. The values of h and s are given in two columns of our text but the third column gives not l but h^2/l^2 with successive entries decreasing almost linearly while there is great variation in the other columns. He discusses the contents of this tablet at great length and is convincing in suggesting that the following Euclidean relations (*Elements*, book 10, prob. 28, lemma) were known more than a thousand years before Pythagoras.

$$l = 2pq, \qquad s = p^2 - q^2, \qquad h = p^2 + q^2,$$

p and q being relatively prime, and $p > q$. Also that the following relation (if $\bar{q} = 1/q$ and $\bar{p} = 1/p$) was not only known:

$$h/l = \tfrac{1}{2}(p\bar{q} + q\bar{p}),$$

but was the basis, with known tables of reciprocals, for the discovery of the successive entries of the tablet. In 13 of the 15 triangles the sides are relatively prime. The lengths of the sides of the largest triangle are:

650 700, 649 909, 1 080 541.

Was EUCLID indebted to Babylonians for his tenth book lemma? As yet no cuneiform mathematical tablets date from the period 1300 to 300 B.C. so that the origins of later Babylonian results are mostly unknown.

Numerous problems involving portions of a right triangle cut off by lines parallel to a side lead to systems of simultaneous equations, even as many as ten equations in ten unknowns; and also to the solution of quadratic equations. Moreover the Babylonian of 1800 B.C. evidently knew our formula for the solution of a quadratic equation with the positive sign before the radical. Many problems could be cited to prove this [14, 16, 17, 21]; let us consider one of them on a tablet in Strassburg dating from about 1800 B.C.

"An area [A] (consisting of) the sum of two squares [x^2+y^2] (is) 16, 40 [$=1000$]." The side of one square [y] (is) $\frac{2}{3}\left[\frac{\alpha}{\beta}\right]$ of the side of the other square [x], diminished by 10[d]. "What are the sides of the squares [x, and y]?"

Hence

$$x^2 + y^2 = A, \qquad y = \frac{\alpha}{\beta}x - d.$$

If $\zeta = x/\beta$, we get a quadratic equation:

$$(\alpha^2 + \beta^2)\zeta^2 - 2d\alpha\zeta = A - d^2,$$

the solution being

$$\zeta = \frac{1}{\alpha^2 + \beta^2}\left[d\alpha \pm \sqrt{\{d^2\alpha^2 + (\alpha^2 + \beta^2)(A - d^2)\}}\right].$$

Bearing in mind that, in the particular case, $A = 1000$, $d = 10$, $\alpha = 40$, $\beta = 60$, let us now compare with this the working given in the text, with our numbers substituted for the Babylonian, of which a few samples are given.

"You proceed thus:	
Square 10: this gives [1, 40] 100; subtract [1, 40] 100 from [16, 40] 1000: this gives [15, 0]900.	$A - d^2 = 900.$
Square [1, 0]60: this gives [1, 0, 0] 3600; 40^2 is [25, 40] 1600: 3600+1600=5200.	$\alpha^2 + \beta^2 = 5200.$
Multiply 5200 by 900: this gives 4680000.	$(\alpha^2 + \beta^2)(A - d^2) = 4680000.$
Multiply 40 by 10: this gives 400.	$\alpha d = 400.$
Square 400: this gives 160000.	$\alpha^2 d^2 = 160000.$
Add 160000 to 4680000: this gives 4840000.	$\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2) = 4840000.$
The square root of this is 2200.	$\sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2200.$
Add 400 already found: this gives 2600.	$d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2600.$
What part of 5200 gives 2600? Answer: one-half (30 in text).	$\dfrac{d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}}}{\alpha^2 + \beta^2}\,[= \zeta] = \frac{1}{2}.$
$\frac{1}{2}$ multiplied by 60 gives 30 as [side of] greater square.	$\frac{1}{2}\beta = \zeta\beta = x = 30.$
Multiply $\frac{1}{2}$ by 40: this gives 20.	$\frac{1}{2}\alpha = \zeta\alpha = 20.$
Subtract 10 from 20 and this gives 10 as [side of] lesser square."	$\zeta\alpha - d = y = 10.$

Surely such work "is wonderful in itself; it is equally extraordinary that these developments in arithmetic and algebra should have remained, for the most of 1800 years at all events, unknown to, or at least without (so far as we can judge) any traceable effect upon, the Greek pioneers in the same subject" [22].

Another interesting method of solution of quadratic equations may be noted. On a Louvre tablet of about 300 B.C. are four problems [23] concerning rectangles of unit area but with the sum of adjacent sides varying $x+y=a$, $xy=1$. The successive steps of the solution are equivalent to substitution in the formula

$$\tfrac{1}{2}(x - y) = \pm\left\{\left[\tfrac{1}{2}(x + y)\right]^2 - xy\right\}^{1/2}, \quad \text{or}$$

$$\left[\tfrac{1}{2}(x - y)\right]^2 = \left[\tfrac{1}{2}(x + y)\right]^2 - xy \quad \text{(EUCLID'S } \textit{Elements}\text{, II, 5 and } \textit{Data}\text{, prop. 85).}$$

In a Yale text of about 1700 B.C. this same method of solution is applied [24] to the equations $x-y=7$, $xy=1$, 0. By this method also the solution of somewhat more complicated rectangle problems on Strassburg and Louvre tablets, [25] of about 1800 B.C., would follow at once.

Our present knowledge of Babylonian achievements in mathematics is mainly due to the extraordinary discoveries of OTTO NEUGEBAUER, a research professor and chairman of the History of Mathematics Department of Brown University. (Dr. SACHS is also a member of this Department.) He discovered not only most of the results set forth above, but also many other things; in particular that the Babylonians discussed problems involving cubic [26] and biquadratic equations. A tablet was discovered which gives not only the squares and cubes of all integers from 1 to 30, but also the results for the sum n^3+n^2 for this same range. Pure cubic equations with integral solutions could be solved by means of the table of cubes reversed, and an example of this kind is quoted. The use of tables of cubes was not previously surmised. Another problem, of about 1800 B.C., seems to call for the solution of simultaneous equations, $xyz+xy=1\frac{1}{6}$, $y=\alpha x$, $z=\mu x$, $(\alpha=2/3,\ \mu=12)$, which lead to $(\mu x)^3+(\mu x)^2=252$. The solution of this equation may be found from the n^3+n^2 table. Two problems lead to such equations. Another problem, of the same period, seems to lead to the general equation $\mu x^3+(1-\mu b)x^2-bx+a=0$, being derived from $xyz+xy=a$, $z=\mu x$, $x+y=b(a=7/6,\ b=5/6,\ \mu=12)$. But in the tablet it is stated that $\frac{xyz+xy}{\mu b^3}=\frac{x}{b}\cdot\frac{y}{b}\cdot\frac{z+1}{\mu b}=\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{7}{10}$ whence it is inferred that $\frac{x}{b}=\frac{3}{5}$, $\frac{y}{b}=\frac{2}{5}$, $\frac{z+1}{\mu b}=\frac{7}{10}$. Thus, this general cubic equation is not solved by reduction to the "normal form," $n^3+n^2=c$, to which, however, NEUGEBAUER believes that the Babylonians were quite capable of reducing the general cubic equation, although as yet he has no evidence that they actually did do it. In connection with the n^3+n^2 table NEUGEBAUER noted also [27] that they may well have known the equivalent of the relation $\sum_{i=1}^{n} i^3=(\sum_{i=1}^{n} i)^2$ for various values of n.

Tablets at Yale University [16] containing hundreds of problems, without solutions, arranged in systematic order, are of great interest. Possibly these may date from 1600 B.C. In several cases of simultaneous equations for solution and leading to biquadratic equations, we find here a number of extraordinary examples of negative numbers in right-hand members. For instance, solve the equations

$$xy = 600 \quad \text{and} \quad 150(x-y)-(x+y)^2=-1000.$$

Babylonian astronomical texts of the third century B.C. [28] make explicit use of the rules, $+\times+=-\times-=+$;
$+\times-=-\times+=-$.

On a Berlin tablet closely related to Yale tablets, the equations (also leading

to a biquadratic equation)

$$xy = 600 \quad \text{and} \quad x^2 + \frac{1}{13}\left\{\frac{1}{19}\left[(x+y)^2 - 600\right] + 3y^2\right\} = 1000,$$

are especially interesting because of the "irregular" numbers in the fractions 1/13 and 1/19. All problems which have so far been found to call for division by such numbers as 7, 11, and 13, are arranged in such a way that the divisors will disappear in the course of the work.

Two other illustrations from the Yale tablets may be cited. A general cubic equation comes up in the discussion of volumes of frustums of a pyramid, as the result of eliminating z from equations of the type, $z(x^2+y^2)=A$, $z=(ay+b)$, $x=c$. An equation of the sixth degree (equivalent to a quadratic in x^3) results from the solution of equations of the form $xy=b$, $a_1x^2/y+a_2y^2/x+a_3=0$.

A sexagesimal number, n, is regular if its reciprocal, $\bar{n}$, is a finite sexagesimal expression, and irregular if it is not. The necessary and sufficient condition for n to be regular is that $n=2^\alpha 3^\beta 5^\gamma$, where α, β, γ are each a positive integer or zero. With the exception of a Yale tablet, tables of reciprocals contain only reciprocals of regular numbers. For an irregular number like 7 it might be thought that such approximations as 7/48 or 13/90 might be found, but this is not the case. On the unique tablet to which we have just referred are such approximations as the following:

$$\overline{59} = ; 1, 1, 1 \qquad \overline{61} = ; 0, 59, 0, 59 \quad \text{and} \quad \overline{78} = ; 0, 56, 9, 13, 50.$$

The Babylonian method of finding the reciprocal of any regular number, however complicated, is now well known [29]. For such a number c the basic relation is $\bar{c}=\bar{a}(\overline{1+b\bar{a}})$, where a and b are two numbers such that $c=a+b$, and a is a number whose reciprocal may be found in a standard table.

In Babylonian mathematics tables were constantly used, and in particular, tables of reciprocals, reducing division to multiplication, but these tables rarely go beyond two sexagesimal places (3600). There is, however, a Louvre text of about 300 B.C. with scores of larger entries and including one of an n, as a seven-place number (corresponding to eleven decimal places), leading to $\bar{n}$, a seventeen-place number (corresponding to twenty-nine decimal places). Such tables were necessary in the Babylonian astronomical calculations [30] of the time. Thus there was mathematical development to meet astronomical needs.

On another Louvre tablet about the time of ARCHIMEDES, NEUGEBAUER found two other suggestive problems [32]. One states that

$$1 + 2 + 2^2 + \cdots + 2^9 = 2^9 + 2^9 - 1.$$

Is this another way of writing $2\cdot 2^9-1=(2^{10}-1)/(2-1)$, indicating knowledge of EUCLID'S, or our own, formula for the sum of such a geometric series?

On the same tablet it is stated that

$$1\cdot 1 + 2\cdot 2 + 3\cdot 3 + \cdots + 10\cdot 10 = (1\cdot\tfrac{1}{3} + 10\cdot\tfrac{2}{3})\cdot 55 = 385.$$

Now $\sum_{i=1}^{n} i^2 = (1\cdot\frac{1}{3} + n\cdot\frac{2}{3})\cdot\sum_{i=1}^{n} i = \frac{1}{6}n(n+1)(2n+1)$, if we set $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, a result known to the PYTHAGOREANS. This sum-of-squares formula was practically equivalent to one known to ARCHIMEDES. Did contemporary Babylonians also use it?

Istanbul tablets seem originally to have had tables of c^n, for $n=1$ to 10, for $c=9$, $c=16$, $c=100$, and $c=225$. An application for such a table would be in solving exponential equations of the type $a^x=b$. Such equations arose in work of the Babylonian, even in the case where x was not integral, as one may check by a Louvre tablet of about 1700 B.C. One problem here is to find how long it would take for a certain sum of money to double itself at 20% interest. The problem is, then, to find x, the number of years, in the equation $(1;12)^x=2$. $1;12^3<2<1;12^4$, hence $x>3$ but <4. Linear interpolation gives

$$\frac{1;12^4 - 2}{1;12^4 - 1;12^3} = 0;12,46,40 \text{ years};$$

or 2;33,20 months. That is, $x=4$ years $-2;33,20$ months.

Since the final result thus found is exactly what is given in the tablet, the method used in reaching it, may well have been that employed by the Babylonians. Both in the Berlin Museum and in Yale University are tablets with other problems in compound interest.

Babylonian approximations to the square roots of non-square numbers are of interest; for example, 1 5/12 for $\sqrt{2}$, and 17/24 for $1/\sqrt{2}$; here, and in finding $\sqrt{1700}$, an approximation formula equivalent to one employed by HERON of Alexandria [97] seems to have been used. In finding 2β as approximation to $\sqrt{2\frac{1}{2}}$, the use of the equivalent of a DIOPHANTINE equation is suggested. But the most remarkable approximation is that found in a Yale tablet of about -1600 for the diagonal of the side of a unit square $\sqrt{2}\approx 1;24,51,10$ which is 1.414213 instead of 1.414214. In explaining the possible derivation of this result NEUGEBAUER & SACHS present a strong case for the Babylonians having followed a procedure consisting of alternating approximation of $\sqrt{2}$ by arithmetic and harmonic means of approximations previously found [33].

The brief suggestions which we have given of discussions of purely numerical problems of very varied types, could be greatly magnified, and force upon us the conclusion that Babylonians during 2000 years before the Christian era laid the foundations of real algebra. Even where the foundation is apparently geometric the essence is usually strongly algebraic, illustrated by the fact that frequently operations occur which do not admit of a geometric interpretation, like the addition of areas and lengths, or multiplication of areas. The predominant problem consists in the determination of unknown quantities subject to given conditions. Of course a certain number of geometric relations were well known. Numerical calculations are everywhere carried out with the greatest facility and skill.

The criterion for scientific mathematics must be the existence of the concept of proof. Egyptian mathematics contains only two (rather trivial) general

rules (RHIND papyrus [43, 44], problems 61, 66), and the much more highly developed Babylonian mathematics displays substantial illustrations of general techniques for proving its procedures [34]. But the remarkable mathematical astronomy which flourished during the last three centuries preceding the Christian era, and had as its goal the computation of ephemerides for the moon and planets, was distinctly scientific [27, 28].

Extension of our knowledge in connection with Babylonian mathematics and astronomy is likely long to continue. Let us now consider achievements of the Egyptians in pure and applied mathematics [35], where recent research has contributed little that is new.

We have referred to the mace record [36] of 3100 B.C. The erection of the great pyramid at Gizeh about 2900 B.C. must have involved many mathematical problems. Here was a huge structure covering more than 13 acres which, with its marvelous connecting roadway to the Nile, took 100,000 workmen 30 years to build. Over 2,000,000 blocks of stone, averaging $2\frac{1}{2}$ tons in weight, and fitted together with great exactness were brought from sandstone quarries on the opposite side of the Nile. For the roofs of chambers granite blocks 27 feet long and 4 feet thick, weighing 54 tons each, were transported from the quarry more than 600 miles away, and placed in their position over 200 feet above the ground level [37].

But the problems of mechanics and engineering involved in handling even the large stone blocks of the great pyramid were slight as compared with those dealt with by the Egyptians in quarrying and setting up some of their huge obelisks of pink granite. The largest existing obelisk, quarried about 1500 B.C., was no less than 105 feet long, nearly 10 feet square at the larger end, and about 430 tons in weight. It was set up in front of the Temple of the Sun at Thebes [38].

A certain amount of mathematics was used by the Ancient Egyptians in connection with their devising of timepieces, namely: (a) the sun-dial or gnomon; and (b) the clepsydra or water-clock [39]. The same may be said concerning ship-building and navigation [40] and engineering involved in the building of notable aqueducts and tunnels for providing water supplies [41].

Most of our knowledge of Egyptian mathematics is derived from two mathematical papyri, the one written about 1850 B.C., usually called the Moscow papyrus [42], containing 25 problems; and the other dating from about 1650 B.C., commonly called the RHIND mathematical papyrus [43, 44], with 85 problems. All of the 110 problems are numerical and many of them are excessively simple. Those in the RHIND papyrus are preceded by a table giving the equivalents in unit fractions of 2 divided by all odd numbers from 5 to 101; for with the exception of $\frac{2}{3}$ the Egyptian had no notation for a fraction with a numerator other than unity. Two divided by 7 is expressed as the sum of $\frac{1}{4}$ and 1/28; 2 divided by 97 is expressed as the sum of 1/56, 1/679, 1/776. Both of these results, as well as $\frac{2}{3}$ expressed as the sum of $\frac{1}{2}$ and $\frac{1}{6}$, are used in a single problem later. The table was therefore a reference list for use in solving problems.

The Egyptian carried through the multiplication of two numbers by successive multiplication of one of the numbers by twos or tens or by $\frac{2}{3}$, or by the division of the number by twos or tens. A sort of transposition was also a common operation. For example, if it had been found that $\frac{1}{5}$ of 105 was 21 the Egyptian might at once write down 1/21 of 105 is 5. Problems in division are reduced to those of multiplication. It is rather extraordinary that in order to get *one*-third of a number, the Egyptian first found *two*-thirds of the number and then took one-half of the result. This is illustrated in more than a dozen problems of the RHIND papyrus.

Nearly a score of 110 problems are such as we would now solve by algebra with equations of the first degree in one unknown quantity. For example: "A quantity, its $\frac{2}{3}$, its $\frac{1}{2}$, and its $\frac{1}{7}$, added together, becomes 33. What is the quantity?" The method of solution used is generally that of trial, or false position, and in more than one case it is obvious that the idea of proportion was clearly understood.

Another score of the problems deal with such questions as the strength of bread and of different kinds of beer, the derivation of beer of great alcoholic strength from two others, and the exchange of beer for bread. An example of this type of problem is the following: "Given that 13 hekat of upper Egyptian grain is made into 18 des of besha date-substitute beer, and that 1 des of this makes $2\frac{1}{2}$ des of barley beer, what is the strength of the barley beer?"

The feed for geese, cranes, ducks, quails, doves, and also for bulls and common cattle is discussed in other problems. The following illustrates a rule-of-three problem: "A sandal maker works for 15 days receiving wages every 5 days. If he does the work in 10 days after what periods should he be paid?" Of problems in arithmetic progressions the following may be mentioned: "Divide 100 loaves among 5 men in such a way that the share received shall be in arithmetic progression and that one-seventh of the sum of the largest three shares shall be equal to the sum of the smallest two." Remember that these are problems of 1650 to 1850 B.C.

Twenty-six of the 110 problems are geometric [**45**], and both volumes and areas are discussed. The area of a circle is repeatedly taken as the square of 8/9 of the diameter; this leads us to the remarkable value $256/81 = 3.1605 \cdots$ for π, much better than the somewhat earlier Babylonian value 3. The volume of a right circular cylindric granary is taken as equal to the area of its base multiplied by the number of units in its height. The most recent discussion seems to make it clear that the Egyptian knew that in any triangle its area is equal to one-half the product of its base and altitude [**45**]. The cotangents of the angles which the faces of pyramids make with their bases are discussed. A numerical problem appears to prove the extraordinary fact that the Egyptian knew our formula for the volume of a frustum of a square pyramid $V = (h/3)(a^2+ab+b^2)$, where a and b are the lengths of the sides of the square bases and h is the number of height units of the frustum. The editor of the Moscow papyrus, which was first completely published in 1930 [**42**], believed that yet another problem gave

the correct result for the area of a hemisphere, if we take the value of π to which reference was just made. But the late Professor PEET, a prominent English Egyptologist, argued that there was no proper ground for this conclusion, since the correct translation led to something quite different [45].

There is no document to prove that the Egyptian knew even a particular case of the PYTHAGOREAN theorem [43].

Our main sources of information concerning Egyptian mathematics consist of two papyri of much the same type, but all additional fragments which we possess match the same picture, which is paralleled by economic documents in which occur precisely those problems and methods which we find in the mathematical papyri. Furthermore, the Egyptian mathematical texts find their continuation in Greek papyri (on to the eighth century of our era), which again show the same pattern. It is therefore safe to say that Egyptian mathematics never rose above a very primitive level, and did not provide the most essential tools for astronomical computation of real importance [30].

Thus we conclude our consideration of the mathematics of the Babylonians and Egyptians in our first period. Before 600 B.C. there was no other mathematics than theirs worth considering.

B. GREEK MATHEMATICS 600 B.C. TO 600 A.D. [46–51]

Greek history began with the second millennium B.C. but in connection with the history of mathematics, of sculpture, of architecture, of art, of philosophy, of literature, of thought, the semi-millennium commencing about 600 B.C. is of the greatest importance. The Greeks were not confined to the Greek peninsula, as in modern times. They occupied Macedonia and Thrace, the islands of the Aegean, the northern and western seaboards of Asia Minor, Southern Italy and Sicily. Scattered settlements were also to be found as far apart as the mouth of the Rhone, the north of Africa, and the eastern end of the Black Sea. Such was the location of the people who were presently to set such marvelous everlasting beacon lights of Freedom, Truth, Reason, Beauty, Excellence, Fellowship between man and man, which were to inspire, guide, and sustain through the millenniums to follow.

What were the special aptitudes which the Greeks possessed for science? An answer by SIR THOMAS HEATH [48], long a leading authority on Greek mathematics, is given. But into this quotation two correcting interpolations are introduced.

"They had, first, a love of knowledge for its own sake, amounting, as Butcher says, to an instinct and a passion; secondly, a love of truth and a determination to see things as they are; thirdly, a remarkable capacity for accurate observation." [In "capacity for accurate observation" there seems to be no difference between Greek and other cultures. The representation of animals by Egyptians and Assyrians did not find their equal in Greece; because of their exactness they are constantly used by zoologists as source material. The astronomical observations of the Babylonians are certainly not of less importance

than those of the Greeks, and also the fundamental instruments such as gnomon, sundial, water-clock. Ancient Egyptian anatomy was of the highest order.] "Fourthly, while eagerly assimilating information from all quarters, from Egypt and Babylon in particular, they had an unerring instinct for taking what was worth having and rejecting the rest. As one writer has said, 'it remains their everlasting glory that they discovered and made use of the serious scientific elements in the confused and complex mass of exact observations and superstitious ideas which constitutes the priestly wisdom of the East, and threw all the fantastic rubbish on one side.'" ["Rejecting the rest" and "Superstitious ideas . . . rubbish on one side." These stand in contradiction to many historical facts. All forms of oriental culture found an entry into Greece (see, for example, R. REITZENSTEIN, *Die hellenistischen Mysterienreligionen nach ihren Grundgedanken und Wirkungen*, third ed., Leipzig, 1927.) Astrology in its absurd forms was first a product of Hellenism.] "Fifthly, they possessed a speculative genius unrivaled in the world's history.

"It was this unique combination of gifts which qualified the Greeks to lead the world in all the intellectual pursuits that make life worth living.

"Last, but not less important, the Greeks possessed the advantage over the Egyptians and Babylonians of having no priesthood which could monopolise learning as a preserve of its own, with the inevitable result of sterilising it by keeping it bound up with religious dogmas and prescribed and narrow routine."

Some of these attributes were possibly more in evidence in their great achievements in the fields of medicine, biology, and natural science, than in those of mathematics and astronomy, which now concern us for a few moments. Since hours would be necessary for any adequate description of their wonderful achievements in these fields, we must confine ourselves largely to references to a few names and results.

Greek theoretical geometry and astronomy began with **Thales** [51] of Miletus, on the West coast of Asia Minor, in the first half of the sixth century B.C. No wonder he was declared one of the Seven Wise Men since, apart from being a mathematician and astronomer, he was also a statesman, engineer, man of business, and philosopher. To THALES later tradition attributes the following results in elementary geometry, all of them in EUCLID'S *Elements:*

1. A circle is bisected by any diameter (I, def. 17);
2. The angles at the base of an isosceles triangle are equal (I, 5);
3. If two straight lines cut one another, the vertically opposite angles are respectively equal (I, 15);
4. If two triangles have two angles and one side in each respectively equal the triangles are equal in all respects (I, 26);
5. The angle in a semi-circle is a right angle (which we have seen, was already recognized by the Babylonians [55] some 1400 years earlier; III, 31).

Of practical problems he showed how to determine the distance of a ship from the shore and found the height of a pyramid by means of the shadows cast on

the ground at the same moment by the pyramid and a stick; that moment was chosen when the length of the stick and its shadow were equal. There is no evidence that Thales predicted a solar eclipse which took place in 585 B.C. (See A. PANNEKOEK, "The origin of the Saros," K. Akad. van Wetens., Amsterdam, *Proc.*, 1918, v. 20, p. 955.) It may be of interest to remark that THALES was the first known individual with whom definite mathematical discoveries were associated.

About half a century after THALES came **PYTHAGORAS** [**52**]. Under his inspiration geometry was first pursued as a study for its own sake. A man of great ability and a most interesting and magnetic mystic, he finally settled at Crotona on the southeastern coast of Italy. Here among the young men of well-to-do families he established a secret society or brotherhood most of whose mathematical discoveries were pooled. According to HEATH [**22**], it is to the PYTHAGOREANS, that is about 500–350 B.C., that the following geometric results are due:

1. The properties of parallels and their application to prove that the sum of the angles of a triangle is equal to two right angles. From this were deduced the familiar results concerning the sums of (*a*) exterior and (*b*) interior angles of a polygon.
2. The transformation of areas of rectilinear figures, and the sums and differences of such areas, into equivalent areas of different shapes. To this end they invented the powerful method of application of areas, the main constituent of the geometric algebra by which they effected the geometric equivalent of addition, subtraction, division, extraction of the square root, and the complete solution of the general quadratic equation $x^2 \pm px \pm q = 0$, so far as it has real roots.

At a considerably later day APOLLONIUS of Perga named the conic sections, parabola, ellipse, and hyperbola, because he had shown that these curves were respectively defined by the application of an area, the application of an area falling short, the application of an area exceeding; that is,

$$y^2 = px, \quad y^2 = px - \frac{p}{d}x^2, \quad \text{and} \quad y^2 = px + \frac{p}{d}x^2.$$

3. The PYTHAGOREANS had a theory of proportion pretty fully developed, though it was only applicable to commensurable magnitudes, being presumably a numerical theory. They knew properties of similar figures such as similar rectilinear figures being in the duplicate ratio of corresponding sides.
4. They had discovered, or were aware of the existence of, at least three of the regular solids—the tetrahedron, cube, and dodecahedron.
5. They discovered the existence of the incommensurable in at least one case, that of the diagonal of a square in relation to its side, and they devised a method of obtaining closer and closer approximations to the value of $\sqrt{2}$ in the form of numerical fractions, x/y whose elements

are the successive solutions of the indeterminate equations $x^2-2y^2=\pm 1$; $[y=2, 5, 12, 29, \cdots; x=3, 7, 17, 41, \cdots]$.

Arithmetic in the sense of the theory of numbers began in discussions connected with problems of the PYTHAGOREANS. With the Greeks, Arithmetic dealing with absolute numbers, or numbers in the abstract, was distinguished from Logistic [50], the science dealing with ordinary arithmetical operations, and certain problems of elementary algebra.

Two numbers are called amicable or friendly if each equals the sum of the aliquot divisors of the other. PYTHAGORAS gave the first pair 220, 284 [56]. The second and third pairs were given in the seventeenth century by FERMAT and DESCARTES respectively, and the next 61 pairs were obtained by Euler in the eighteenth century. PYTHAGOREANS discussed also various forms of figured numbers—triangular, square, pentagonal, *etc.*, the number of dots in a figure corresponding to the number. Thus $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$ is any triangular number; $1+3+5+7+\cdots+(2n-1)=n^2$ is any square number and so on. To them is also attributed the formula $m^2+[\frac{1}{2}(m^2-1)]^2=[\frac{1}{2}(m^2+1)]^2$, where m was any odd number. But their most wonderful discovery about numbers was in showing the dependence of musical intervals upon numerical ratios. They found that for strings of the same tension, the different lengths of strings would be in the ratio of 2 to 1 for the octave, 3 to 2 for the fifth, and 4 to 3 for the fourth. They went on to form a diatonic scale and thus initiated a study which was to be extensively elaborated by later Greeks [57]. The PYTHAGOREANS showed interest in astronomy, associating musical notes with the heavenly bodies (the highest with the fixed stars, the next highest with Saturn, the lowest with the Moon) and also various numbers [58]. The planets were thought as moving in independent circles. There is no ground for thinking that they regarded the earth as spherical [59].

Thus geometry, arithmetic, music, and astronomy came to be grouped together by the PYTHAGOREANS as fundamental liberal arts for study and to form what was, in the middle ages, called the *quadrivium*. The word "mathematics" is derived from a Greek word *μάθημα* meaning simply a "subject of instruction"; but by the time of ARISTOTLE [60] (−340) the term was definitely restricted to subjects of the *quadrivium*.

PLATO [61, 62] regarded mathematics in its four branches, arithmetic, geometry, stereometry, and astronomy, as the first essential in the training of philosophers and of those who should rule his ideal State; "let no one destitute of geometry enter my doors," said the inscription over the door of his school near Athens. He emphasized that mathematics is of value for the training of the mind, and that by comparison, its practical value is of no account. It is not known that PLATO made any mathematical discovery. THEAETETUS, pupil of SOCRATES and friend of PLATO, one of whose dialogues, *Theaetetus*, is a commemorative tribute, advanced the theory of irrationals and was probably the first to construct all five of the regular solids [62] theoretically, and to in-

vestigate fully their relations to one another, and to the circumscribed sphere as elaborated in EUCLID's *Elements*, XIII; indeed THEAETETUS was the discoverer of the octahedron and icosahedron. In the fifth century B.C. flourished **ZENO** of Elea [63] who also does not seem to have been a mathematician, yet his paradoxes of motion had a profound influence on the course which geometry thereafter took.

The PYTHAGOREAN difficulties in connection with geometric proportion and incommensurables were completely solved in the fourth century B.C. by a pupil of ARCHYTAS and PLATO named **EUDOXUS** [64], born at Cnidos on the west coast of Asia Minor. He was an original genius second only to ARCHIMEDES. His masterly setting forth of incommensurables, as in the fifth book of EUCLID's *Elements*, is practically identical with the modern formulation of DEDEKIND. EUDOXUS discovered also the so-called "method of exhaustion" by means of which he gave the first rigorous proof of the results for the volume of a cone, and of a pyramid, and probably showed that: (1) the areas of two circles are to one another as the squares of their diameters; and (2) the volumes of spheres are to one another as the cubes of the lengths of their respective diameters.

It seems certain that **MENAECHMUS**, a pupil of EUDOXUS, was the discoverer of the conic sections—parabola, ellipse, hyperbola, which were originally thought of as sections perpendicular to generators of right-angled, acute-angled, and obtuse-angled cones; see NEUGEBAUER, 1948, [86]. He showed that by means of the intersection of two parabolas, or of a parabola and a rectangular hyperbola so obtained, we could find two mean proportionals between two lines of lengths a and b. This implies the recognition of a geometric relation, in the case of the parabola, equivalent to our ordinary Cartesian equation, and of a similar relation for a rectangular hyperbola referred to its asymptotes, as axes.

This discovery of MENAECHMUS was of particular interest because it gave a new solution of the problem of the duplication of the cube, one of the three famous problems which had been formulated in the fifth century B.C. These problems are [65–69].

(*a*) To find a line which shall be the edge of a cube whose volume is double that of a given cube, the problem of the duplication of the cube.
(*b*) To trisect any given angle.
(*c*) To find a line which shall be the side of a square whose area shall be exactly equal to that of a given circle, the problem of squaring the circle [67].

All of these problems were solved by the Greeks within a century, but more than 22 centuries were to pass before it was finally proved that no one of them could be solved with ruler and compasses alone. As early as the fifth century B.C., by means of a curve called the quadratrix [69], the problem of the trisection of an angle was solved by **HIPPIAS** of Elis, and **HIPPOCRATES** of Chios had shown that the problem of duplicating the cube could be reduced to that of finding two mean proportionals between such lengths as a and $2a$. Such mean

proportionals were found by ARCHYTAS, a PYTHAGOREAN, a friend of PLATO, a statesman and philosopher, in a marvellous construction by means of the intersection of a right cone, a cylinder, and an anchor ring with inner diameter zero. DINOSTRATUS, a brother of MENAECHMUS, showed that the quadratrix of HIPPIAS could also be used to solve the problem of squaring the circle. In the third century this problem was also solved, in effect, by a spiral invented by ARCHIMEDES [69]. In the third century a quartic curve, the conchoid of NICOMEDES [69], was used to solve the problems of trisecting an angle, and duplicating a cube, and the cissoid of DIOCLES [69], a cubic curve, was also employed for solving the latter problem.

We come now to the consideration of the golden period of Greek mathematics and of the greatest mathematical school of ancient times. This was at the magnificently endowed university of Alexandria which had been founded by ALEXANDER the Great in 332 B.C. [70]. The university was opened about 300 B.C. and within the first 40 years of its existence over 600,000 rolls had been collected in its great library. EUCLID [71] was a professor of mathematics in the university. Practically nothing is known about his life except that he was the author of at least ten treatises [72] of which approximately complete texts of five are available. These include three on applied mathematics, namely on phaenomena [75], on optics [76], and on music [77]. But by far the most famous one is his treatise, in 13 books, called the Elements [73, 74]. More than a thousand editions have appeared since the first one printed in 1482, and for 1800 years before that, manuscript copies dominated all teaching of geometry. Though a large portion of the subject matter had been investigated by predecessors the whole arrangement was due to the great genius of EUCLID who supplied innumerable details. It is quite impossible to give in a few sentences any adequate idea of the contents of the 465 propositions in this monumental work. Practically all of the geometric material of American school texts in plane and solid geometry is contained in parts of six of the books (1, 3, 4, 6, 11 and 12) of the Elements. Most books are prefaced by definitions, but before the first book are certain postulates the fifth of which is the famous one which differentiates Euclidean from noneuclidean geometry [78]. The statement is as follows: "Let it be postulated that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which are the angles less than two right angles."

In Book II of the Elements are a number of propositions giving geometric proofs of algebraic identities, such as $(a+b)^2=a^2+2ab+b^2$, and in Books II and VI are the propositions giving, among many other things, geometric solutions of quadratic equations [98], which were treated algebraically by the Babylonians. In Book V is expounded the remarkable EUDOXUS theory of proportion, alike applicable to incommensurable and commensurable magnitudes of every kind. There are many masterly Euclidean developments. Book VI applies to plane geometry the general theory of proportion set out in Book V. The 102 proposi-

tions in Books VII, VIII, IX deal with questions in the theory of numbers. Special reference may be made to three propositions in Book IX. It is elegantly proved in the 20th that there are an infinite number of prime numbers; in the 35th there is a beautiful geometric derivation of what is practically equivalent to our algebraic formula for the sum of the first n terms of a geometric progression; and in the 36th it is proved that if $S_n = 2^n - 1$ is prime, then $P = 2^{n-1}S_n$ is a perfect number [79], that is, a number which is equal to the sum of its divisors smaller than itself. The first four perfect numbers are 6, 28, 496, and 8128; only 12 perfect numbers are known, and they are all of the EUCLIDEAN type.

Book X contains 115 propositions on incommensurables, and is perhaps the most remarkable and most finished of all the books of the Elements. Books XI, XII, XIII deal with geometry of three dimensions. In propositions 16 and 17 of the last book all the details are carried out for the actual construction of an icosahedron, and of a dodecahedron, inscribed in a sphere.

The logical form of the presentation of the propositions in the Elements is especially notable. There is first of all the enunciation of the proposition; secondly, the statement of the precise data; thirdly, the statement of what we are required to do with reference to the precise data mentioned; fourthly, the construction, the addition when necessary of more lines to the figure; fifthly, the proof; and sixthly, the conclusion, stating what has actually been done, which generally follows the wording of the original enunciation.

In the period between EUCLID and ARCHIMEDES came **ARISTARCHUS** of Samos [58], whose great achievement lies in the fact that he was the first to assert that the earth and the other planets (Venus, Mercury, Mars, Jupiter, and Saturn) revolved about the sun, thus anticipating COPERNICUS by seventeen centuries.

ARCHIMEDES [80–82] was born at Syracuse, in Sicily, about 287 B.C. and was killed in the sack of that city by the Romans in 212. He studied with successors of EUCLID at Alexandria and it was probably in Egypt that he invented the water screw, known by his name, for drawing water to irrigate fields. On returning to Syracuse he devoted his time wholly to mathematical research. Up to the time of NEWTON at least he was the greatest mathematical genius that the world had seen. The following summary of his work is mainly due to HEATH [22].

In geometry his works consisted chiefly of original researches in the quadrature of plane curvilinear figures and in the quadrature of curved surfaces and the cubature of solids bounded by such surfaces. By methods equivalent to integration [82], ARCHIMEDES found the area of a parabolic segment, the area of a spiral, the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the solids of revolution of the second degree.

In arithmetic he calculated approximations to the value of π and in the course of these calculations showed that he could find approximate values for the square roots of large or small non-square numbers; he found $3\frac{1}{7} > \pi > 3\frac{10}{71}$, and also, apparently, $3.141697 \cdots > \pi > 3.141495 \cdots$, the mean of which is 3.141596. ARCHIMEDES also invented a system of arithmetic nomenclature by

which he could express in language enormously large numbers, in fact all numbers up to that which we would write as 1 followed by 80,000 million million ciphers.

In mechanics he laid down certain postulates and, on the basis of these postulates, established certain fundamental theorems on magnitudes balancing about a point and on centers of gravity [82], going so far as to find the center of gravity of any segment of a parabola, a semi-circle, a cone, a hemisphere, a segment of a sphere, and a right segment of a paraboloid of revolution. As we learn from his *Method* [81], discovered as recently as 1906, ARCHIMEDES made most ingenious use of mechanics as an aid for suggesting probable results in geometry; the first theorem which he found in this way was in connection with the quadrature of the parabola.

He invented the whole science of hydrostatics, which he carried so far as to give a complete investigation of the positions of rest and stability of a right segment of a paraboloid of revolution floating in a fluid with the base either upwards or downwards, but so that the base is entirely above or entirely below the surface of the fluid. HEATH then sums up as follows:

"The treatises themselves are, without exception, models of mathematical exposition; the gradual unfolding of the plan of attack, the masterly ordering of the propositions, the stern elimination of everything not immediately relevant, the perfect finish of the whole, combine to produce a deep impression, almost a feeling of awe in the mind of the reader."

In accordance with the desire of ARCHIMEDES the figure of a sphere inscribed in a cylinder was engraved on his tomb since he seemed to regard the discovery that the volume of the sphere was two-thirds that of the circumscribed cylinder as his greatest achievement. CICERO found the tomb with this inscription.

Other results due to ARCHIMEDES were the following: (a) our familiar formula for the area of a triangle in terms of its sides; (b) the construction of 14 inscribable semi-regular polyhedra [83]; (c) the solution of a three-term cubic equation, the term of the first degree lacking [80], by means of the intersection of a parabola and a rectangular hyperbola; this occurs in his work on the sphere and cylinder.

The tradition that ARCHIMEDES destroyed Roman ships, by concentrating sun's rays by means of a series of mirrors, was proved by experiment to be entirely feasible [84]. And finally, ARCHIMEDES was much occupied with astronomy and wrote a book on the construction of a sphere so as to imitate the motion in the heavens of the sun, the moon, and the five planets. When CICERO was quaestor in Sicily in 75 B.C., he actually saw the contrivance and gives a description of it.

Almost contemporary with ARCHIMEDES was his friend ERATOSTHENES, who taught in Alexandria and was librarian at the University. To him ARCHIMEDES dedicated his treatise on *Method*. The most famous scientific achievement of ERATOSTHENES was his apparently very accurate determination of the polar circumference of the earth. This was done by observing that when the sun was

in the zenith at Syene, it was exactly 7° 12′ south of the zenith at Alexandria, known to be 5000 stadia distant. According to a recent interpretation [85], an ERATOSTHENES' device, called a sieve, for determining prime numbers, is of considerable interest.

The third great name in connection with the Alexandrian School was APOLLONIUS of Perga [86] who was about 25 years younger than ARCHIMEDES. He was called by his contemporaries the "Great Geometer," because of his extraordinary treatise on Conics, and he was also mentioned as a famous astronomer. He studied at Alexandria. Only seven of the eight books of his Conics have been preserved. The first three seem to correspond to a treatise which EUCLID wrote, but books V, VI, VII contain the discoveries which APOLLONIUS himself had made. To him are due the terms parabola, ellipse, hyperbola, for reasons which we have already explained, and his lines of reference are any diameter and a tangent at its extremity, the cases of these when at right angles being considered only as special cases. Book V treats of various questions concerning normals to conics, and from certain propositions we can readily deduce the CARTESIAN equations to the evolutes of the three conics. Book VII deals mainly with various results in connection with conjugate diameters. There are nearly 400 propositions. The suggestions that I have given are sufficient to indicate that in this work of APOLLONIUS there is far more than is contained in any of our American textbooks on analytical geometry, so far as conic sections are concerned. Foci of central conics are constructed, and that the sum or difference of focal radii of points of such curves is constant, is proved. There is no reference to the focus of the parabola. It seems likely that the focus-directrix property of all three conics was given in EUCLID's Conics, or well known in his time, although it is not explicitly mentioned before PAPPUS, hundreds of years later [86a].

Among a number of other works of APOLLONIUS we shall refer to only three. In his Plane Loci the following familiar results were given: (*a*) If A, B be fixed points, P any other point in the plane, and AP and BP are in a given ratio, the locus of P is a straight line or a circle (circle locus given earlier by ARISTOTLE) according as the given ratio is or is not one of equality; (*b*) If A, B, C, $\cdots$ be any number of fixed points and $\alpha, \beta, \gamma, \cdots$ any constants, the locus of point P, such that $\alpha \cdot AP^2 + \beta \cdot BP^2 + \gamma \cdot CP^2 + \cdots =$ a constant, is a circle.

The second work, on Contacts or Tangencies, has not come down to us, but its principal problem is to describe a circle tangent to three given circles, which is usually known as the problem of APOLLONIUS. Many celebrated mathematicians, such as VIETA, EULER and NEWTON, have worked on this problem, and what is known of the work as a whole has made it the basis of numerous restorations and discussions [87]. The analogous form for spheres was treated synthetically by FERMAT. From the Plane Loci, and the account of Tangencies by PAPPUS, it has been deduced that APOLLONIUS made the following result, in effect, the basis of his solution of the general tangencies problem: The six centers of similitude of three coplanar circles lie by threes on four straight lines [88].

The third work, on Vergings, is also lost, but some of its problems and solu-

tions have been preserved. One of these problems is: Between the side of a given rhombus and its adjacent side produced, insert a straight line of given length and verging to the opposite corner. HUYGENS gave many solutions and NEWTON used the particular case of a square in illustration of discussion [89]. The work of APOLLONIUS in Astronomy was also of importance [89].

We have already seen the marvellous development of geometry by the Greeks from the beginning up to difficult problems of the integral calculus, and all of this within the short space of 350 years.

Possibly contemporary with scientific activity among the Greeks of the fourth and fifth centuries B.C. are two notable Babylonian astronomers [90], **NABURIANU** and **CIDENAS** or **KIDINNU**, who had at their disposal an extraordinary number of Babylonian observations of eclipses for more than three centuries. The next name of mathematical interest among the Greeks is that of their most eminent astronomer, **HIPPARCHUS** [91], who flourished about 140 B.C. From the great mass of data available from about 750 B.C., it was almost inevitable that he should be the first to note the precession of the equinoxes [92]; but it was not till the time of NEWTON that the explanation of this precession made it a matter of real importance. In masterly fashion HIPPARCHUS improved on ARISTARCHUS's calculations of the sizes and distances of the sun and moon, and developed a systematic theory with regard to them. He also compiled a catalogue of 850 stars [93], stating their places and apparent sizes. Trigonometry as a science seems to have begun with him, and also the introduction into Greece of the division of the circle [94] into 360°.

About two centuries after HIPPARCHUS we come to **HERON** [95] of Alexandria whose date, until comparatively recently, has been a matter of uncertainty, even to the extent of 400 years [96]. He was an almost encyclopedic writer on mathematical and physical subjects, and aimed at practical utility rather than theoretical completeness. In his *Pneumatica* are many mechanical devices such as a siphon, a fire engine, a device whereby temple doors are opened by fire on an altar, an altar organ blown by the agency of hand labor or by a windmill, and a jet of steam supporting a sphere [314].

He gave an elegant geometric proof of the formula for the area of a triangle in terms of the sides, now attributed to ARCHIMEDES. His method of approximating to the square root of non-square numbers [97] seems to have been used by early Babylonians. His formula for the volume of the frustum of a square pyramid can be readily reduced to the one used 2000 years earlier in the Moscow papyrus [42].

Quadratic equations are solved in a manner very similar to that of the Babylonians 1900 years before [98]. HERON showed how he obtained the cube root of a non-cube number, 100. He found also the volume of the five regular solids, and of an anchor ring (as given by DIONYSODORUS, a contemporary of APOLLONIUS of Perga). His works on surveying, and the *Dioptra* are of interest, and it is by means of the latter that his date has been narrowed down.

Modern research seems to indicate that **DIOPHANTUS** [99] of Alexandria,

one of the great mathematicians of Greek civilization, should be assigned to the first century of our era [100], rather than to the third century. He was the first to make systematic use of symbols in algebraic work, a sign for unknown, a sign for minus, and signs for the various powers, that is, square, cube, and so on. Of his great work called Arithmetic only 6 of its thirteen books survive and they are mostly taken up with problems in indeterminate analysis of the second degree. Hence the term Diophantine Analysis [101]. The answers are always in positive rational numbers. The collection is extraordinarily varied and the devices resorted to are highly ingenious. In Book V, 9, 11, not only does he find particular solutions of $x^2-26y^2=1$ and $x^2-30y^2=1$, but also shows (VI, 15, lemma) that they have an infinite number of solutions. DIOPHANTUS may have discussed the equation $x^2-Ay^2=1$ more fully in the part of the Arithmetic which has been lost. See later under BHĀSKARA. It has been surmised that the last seven books of the Arithmetic may have been filled with material between the most difficult of DIOPHANTUS' problems and the famous cattle problem of ARCHIMEDES.

A possible contemporary of HERON was **MENELAUS** of Alexandria [102], who wrote a variety of treatises among which his *Sphaerica* [103] contained for the first time the conception and definition of a spherical triangle; here also are the MENELAUS theorem for the sphere, and deductions from it, furnishing the equivalent of formulae in spherical trigonometry.

Much of our knowledge of the achievements of HIPPARCHUS is derived from a work of **CLAUDIUS PTOLEMY** [104] of Alexandria, who flourished in the second century of our era. This great work, of extraordinary compactness and elegance, was called the *Almagest*, which overshadowed all previous works of the kind. In the trigonometry of the early part is derived a table of chords equivalent to a table of sin A for each $15'$ of the quadrant, the values being expressed in parts, minutes and seconds.

In the first book of the *Almagest* is PTOLEMY's theorem regarding a quadrilateral inscribed in a circle, and from this $\sin^2 A+\cos^2 A=1$, and the well-known formulae for sin $(A\pm B)$ and cos $(A\pm B)$, among others, are readily derived [105]. In the fourth book occurs a solution of the so-called "Problem of SNELL" (1617), or of POTHENOT 1692 (publ. 1730): to determine the point from which pairs of three given points are seen under given angles [105].

In this same work results of much of PTOLEMY's observational work are contained in his catalogue of 1028 stars, with the latitude and longitude of each [106]. But his outstanding achievement was to develop a lunar theory and a planetary theory, close to KEPLER's, with planetary movements in eccentric orbits almost elliptical.

PTOLEMY used various projections including stereographic. The essentials of his argument in an attempted proof of EUCLID's parallel postulate have been preserved. It is recorded that he wrote also a work on dimension in which he attempted to prove that the possible number of dimensions is only three.

PTOLEMY wrote also a remarkable geographical treatise with maps, various

later printed editions of which are treasured by libraries [106a]. His work on Optics is especially interesting because it contains the first attempt at a theory of refraction [106b]. He also wrote a notable treatise on music [106c].

And finally we come to PAPPUS [107] of Alexandria who lived at the end of the third century. His great work, entitled *Mathematical Collection*, [108] covers the whole range of Greek geometry and was intended "to be read with the original works (where extant) rather than independently. Where, however, the history of a subject is given, *e.g.*, that of the duplication of the cube, or the finding of the two mean proportionals, the solutions themselves are reproduced, presumably because they were not readily accessible but had to be collected from scattered sources. Even when some accessible classic is being described, the opportunity is taken to give alternative methods or to make improvements in proofs, extensions and so on. Without pretending to great originality, the whole work shows, on the part of the author, a complete grasp over the subjects treated, independence of judgment, and mastery of technique; the style is concise and clear; in short PAPPUS stands out as an accomplished and versatile mathematician and a worthy representative of the classical Greek geometry" [22]. Among the original contributions of PAPPUS are—(*a*) a generalization of the "Pythagorean" theorem to any triangle with certain parallelograms on its sides; (*b*) generalization of the four-line locus to five- or six- or *n*-line loci [86]; (*c*) proof of the invariance of the cross-ratio under a projective transformation; and (*d*) a result often attributed to GULDIN [109], a Swiss mathematician of the seventeenth century: The volume of a solid of revolution is equal to the area of the generating figure, multiplied by the circumference described by the center of gravity of the figure. PAPPUS was also the first to give results possibly known earlier: (*e*) the construction of a conic through five points [86]; and (*f*) the presentation of the focus-directrix property of the three conics [86a].

The fifth book is mainly devoted to the subject of isoperimetry. There is a very interesting passage concerning bees, their orderliness, and the hexagonal form of their cells exactly filling up space about a point [110]. It is later shown that (*a*) a circle is greater than any regular polygon of equal contour; (*b*) a sphere is greater than any of the five regular solids with equal area. The sixth book is mostly astronomical [111], dealing with treatises studied as an introduction to the great *Almagest* of PTOLEMY. There are, however, propositions of mathematical interest including some related to EUCLID's *Optics* [76]. The seventh book, *On the Treasury of Analysis*, is historically the most important of all the books, since it gives an account of a collection of treatises which, after the Elements of EUCLID, provided the body of doctrine necessary for the professional mathematician to know if he was to be regarded as fully equipped for the solution of problems arising in geometry. Of the 12 works listed seven are by APOLLONIUS and three by EUCLID.

THEON [111a] of Alexandria, a fourth century author of an edition of Euclid's Elements, and a commentator on works of EUCLID and PTOLEMY, is a name of importance in the history of mathematics. In the fifth century, PRO-

CLUS, more of a philosopher than a mathematician, placed the historian under great obligation by his commentary on the first book of EUCLID's elements, which is one of our main sources of information on the history of elementary geometry. [313].

In the early part of the sixth century JOHN PHILOPONUS of Alexandria wrote the earliest extant Greek treatise on the Astrolabe [111b], which played an important role in the history of astronomy and of a mathematical projection.

Alexandria was razed by the Arabs in the seventh century. Contrary to the imaginings of some writers, no library of importance was then destroyed, or even in existence there [112].

For the period of history which we have so far been considering, it is to be noted that, in the case of the Babylonians and Egyptians, the actual source material written two to four thousand years ago is available for study. On the other hand, in the case of the Greeks practically not a single source manuscript is dated earlier than a thousand years after the writer flourished. In spite of this great handicap, scholars feel that some reliable complete or partial original texts of works of such writers as EUCLID, ARCHIMEDES, and APOLLONIUS are now available [313].

C. HINDU [113–116], ARABIC, PERSIAN MATHEMATICS 600 to 1200

With PAPPUS creative Greek mathematicians came to an end, and very soon all the great traditions of Greek learning had died out; then followed nearly a thousand years in which slight additions to the sum of mathematical knowledge were made in Europe. Significant contributions came from India and the Arabs.

In this period the Arabian Empire stretched from India to Spain, Bagdad and Cordova being the centers for the reigning caliphs [117]. In extraordinary fashion Arabs at Bagdad developed profound interest in translated Greek mathematical and astronomical works, and also in similar Hindu material. The ninth and tenth centuries may be regarded as the golden age of Arabian mathematicians [118], to whom the world owes a great debt for preserving, and thus making possible the transmission to Europe, of classics in Greek mathematics otherwise lost. A vast field for further investigation in this connection awaits the attention of scholars.

The formula for the area of an inscribed quadrilateral, similar in form to that for a triangle, ascribed to ARCHIMEDES, was first given in the early part of the seventh century by an outstanding Hindu mathematician, named BRAHMAGUPTA [120], but it was not recognized as true for a cyclic figure only. If a, b, c, and d are the lengths of the sides of the quadrilateral, $s=\frac{1}{2}(a+b+c+d)$, and m and n are the lengths of the diagonals, the area is equal to $\sqrt{[(s-a)(s-b)(s-c)(s-d)]}$, $m=\sqrt{[(ab+cd)(ac+bd)/(ad+bc)]}$, and $n=\sqrt{[(ac+bd)(ad+bc)/(ab+cd)]}$. These formulae were considered with a view to determining quadrilaterals whose sides, diagonals, and areas were all rational quantities. In particular BRAHMAGUPTA gave the rule: If $a^2+b^2=c^2$ and $\alpha^2+\beta^2=\gamma^2$, then the

quadrilateral ($a\gamma$, $c\beta$, $b\gamma$, $c\alpha$) is cyclic, the area is rational, and its rational diagonals are at right angles. BHĀSKARA pointed out that similarly rational is such an inscribed quadrilateral, if a pair of adjacent sides are interchanged; he also indicated how to determine the length of the third diagonal, no longer orthogonal to the other diagonal of the new quadrilateral. BRAHMAGUPTA also noted, in terms of any three rational numbers, the formulae for the lengths of sides of an oblique triangle whose altitudes and areas are rational [121].

We have already seen that the PYTHAGOREANS were led to solutions of the equations $x^2-2y^2=\pm 1$, in getting approximations to $\sqrt{2}$. BRAHMAGUPTA and BHĀSKARA [120, 122] (of the twelfth century) gave the method for finding remarkable particular solutions of the equations $x^2-Ay^2=1$, $A=8$, 61, 67, and 92; and the latter found general solutions [123]. BRAHMAGUPTA'S rules lead, in effect, to the general solution of $ax+by=c$, a, b, c integers, and a and b relatively prime, as $x=\pm(cq-bt)$, $y=\pm(-cp+at)$, where t is zero or any integer, and p/q is (in modern phraseology) the second last convergent of a/b, expressed as a continued fraction. These are the most important developments in Hindu mathematics of this period.

Let us now outline the origin of our common numerals [124]. So far as we know the largest number of our numeral forms were first used in India [125]. The 1, 4, and 6 are found in inscriptions of the third century B.C.; the 2, 4, 6, 7, and 9 appear in another inscription about a century later; and the 2, 3, 4, 5, 6, 7, and 9 in caves of the first and second centuries of our era, all in forms which have a considerable resemblance to our own.

The first definite external reference to Hindu numerals is in a note by a bishop who lived in Mesopotamia about 650 A.D. Early in the ninth century the numerals became known to Arabic scholars. Indeed, one of the most prominent of these, MOHAMMED IBN MÛSÀ AL-KHOWÂRIZMÎ [126] (that is, of Khowârizm, now Khiva, in Russian Turkestan) worked at Bagdad, and wrote a treatise on Hindu numeration and arithmetic, which became known to Europeans through a twelfth century Latin translation. The earliest undoubted occurrence of zero in India [127] was in an inscription of 876, in connection with the numbers 50 and 270. But it was used much earlier, indeed by HYPSICLES ($\dot{-}180$) and later Greek astronomers, from whom Hindus probably borrowed the sign. A somewhat similar sign is also in astronomical records of the Mayas of Central America [128], which may date back to the beginning of our era; they had a vigesimal system with the principle of local value. About -400 the Babylonians possessed both the principle of relative local value, and a zero symbol which was used systematically in astronomical texts and computations. In Europe the complete system of numerals with the zero was derived from the Arabs in the twelfth century.

We have referred to the astronomer and mathematician AL-KHOWÂRIZMÎ (the first of several mathematicians often referred to by their place of birth). He wrote a work on algebra with the title "ḥisāb alg̃abr walmuḳābalah," which has been translated "the science of reduction and cancellation." The Arabic

word for reduction, "alg̃abr," thus became our word alg̃ebra. The English word algorism is simply a corruption of AL-KHOWÂRIZMÎ, as in the Latinized title form *Algoritmi di numero indorum*, of AL-KHOWÂRIZMÎ's arithmetic. The algebra contained sections on geometry, quadratic equations solved geometrically [129], and problems of inheritance. But most of AL-KHOWÂRIZMÎ's work was in astronomy, and among 100 of his tables in this field are tables of sines and cotangents [130], the former doubtless indicating Hindu influence. For already in the fifth century Greek trigonometry operating with chords was transformed by the Hindus into consideration of half-chords or sines. Thus we there have a table [131] of 3438 sin A, where the radius = 3438.

From the time of the PYTHAGOREANS, constructions of elementary geometry were imagined as carried through with a ruler and compasses with variable opening. A prominent Arabian mathematician of the tenth century, **ABÛ'L WEFÂ** [132], lectured on geometric constructions with a ruler and compasses with a fixed opening [133]. It was not until centuries later that it was recognized that there was no real limitation here, since it was shown by PONCELET and later by STEINER, in the early part of the nineteenth century, that if a single circle and its center are once drawn in a plane, every construction with ruler and compasses can be carried through with ruler alone.

ABÛ'L WEFÂ contributed notably to the development of trigonometry. Already in his time all six of the trigonometric functions were in use, and he employed our formulae for versed sine, for the tangent and cotangent in terms of sine and cosine, and for the sin of an angle in terms of the sine and cosine of the half angle. PTOLEMY's table of sines for each $\frac{1}{2}°$ was extended by him to each $\frac{1}{4}°$, and he made a similar table for tangents. He found the value of sin 30′ correct to the equivalent of eight places of decimals [134], but his great contributions were in spherical trigonometry where he first used our formulae for a right spherical triangle and also our law of sines formula for an oblique spherical triangle.

In concluding the period under consideration we should make brief reference to a remarkable work of the Persian mathematician, astronomer, poet, **OMAR KHAYYÀM** (that is, OMAR the Tentmaker) [135], known to the western world as the author of the Rubaiyat [136]. He wrote a treatise on algebra in which his geometric treatment of systematically classified cubic equations is central [137]. He obtains a root as the abscissa of a point of intersection of a conic and a circle, or of two conics. Various forms of cubic equations are considered; negative roots are rejected, and not all positive roots are discovered. While we have already noted the solution of cubic equations by MENAECHMUS and ARCHIMEDES it is to be remarked that here the point of view is different; the problem is: How can we solve cubic equations with numerical coefficients? Another notable achievement of OMAR was his correction of the calendar by the introduction of cycles of 33 years. This calendar was more accurate than the one we use today.

In order to round out our survey of Islamic contributions to the develop-

ment of trigonometry we append to the treatment of this period some remarks on contributions of two men of later date. The first is **NASÎR ED-DÎN AL-TÛSÎ** [138], exceptionally able astronomer, mathematician and politician at Bagdad in the thirteenth century. Displaying remarkable scientific thoroughness, and power to integrate and extend earlier discoveries, he wrote the first complete plane and spherical trigonometry independent of any astronomical application [139]. In his discussion the theorem of MENELAUS is basic. All six trigonometric functions are used, and necessary formulae for solving right, and all cases of oblique, spherical triangles are given. Such was the status of trigonometry at the close of the thirteenth century. A recently discovered fourteenth century Arabic commentary by NASÎR ED-DÎN suggests that it contains the first part of OMAR KHAYYÀM'S discussion of difficulties of EUCLID [140]. This part treats material similar to introductory propositions in the noneuclidean geometry of SACCHERI, hundreds of years later.

The second supplementary name is that of **ULUGH BEG** [141], a fifteenth century Persian prince and astronomer, who compiled extraordinary tables of sines and tangents for every minute and correct to the equivalent of eight or ten places of decimals [142]. In calculating the sine table he was led approximately to solve the cubic equation of the angle-trisection problem [143]: given sin 3°, to find sin 1°.

D. EUROPEAN MATHEMATICS 1200 TO 1600

During the period 500 to 1200 the student went to the teacher in the monastery and heard his lectures. But in the thirteenth century universities commenced to spring up at such places as Bologna, Padua, Naples, Paris, Oxford and Cambridge. Scribes making copies of treatises were thus kept busily employed by the universities. By the middle of the fifteenth century, however, their products were being sold as books are today. But such methods of disseminating knowledge were crude when compared with that of the distribution of the printed work. The publication of these with movable type commenced about 1450. More than two hundred mathematical works were printed, in Italy alone, before 1500; but this number was increased to 1527 in the next century.

During three and a half of the four centuries now under consideration Italy made the chief contributions to mathematics, and by far the most outstanding mathematician was one who flourished at the beginning of this period, **LEONARDO** of Pisa [144], often called FIBONACCI, that is, son of BONACCIO. During early life he travelled extensively about the Mediterranean, visiting Egypt, Syria, Greece, Sicily, and southern France, and knowledge thus gleaned regarding arithmetic systems used by merchants of different countries was the basis of a notable work, entitled *Liber Abaci* [145], which he wrote in 1202. This is a storehouse from which for centuries authors got material for works on arithmetic and algebra. The Hindu-Arabic system of numerals was here strongly advocated and illustrated, and the work did much to introduce it into Europe. LEONARDO discussed problems in arithmetic processes, barter, alligation, false

position, square and cube roots. It is in this work that we find the problem: "7 old women went to Rome; each woman had 7 mules; each mule carried 7 sacks; each sack contained 7 loaves; and with each loaf were 7 knives; each knife was put up in 7 sheaths. What is the sum total of all named?" It was this problem which gave the clue to the interpretation of a problem in the RHIND papyrus 2800 years earlier, "In each of 7 houses are 7 cats, each cat kills 7 mice, each mouse would have eaten 7 ears of spelt, and each ear of spelt will produce 7 hekat of grain; how much grain is thereby saved?" The modern conundrum starting out: "As I was going up to St. Ives I met a man with 7 wives, each wife had 7 sacks," etc., further illustrates the perpetuation of this number succession through the centuries. An interesting chapter on similar perpetuation of other problems has been written [146].

In *Liber Abaci* is also the following: How many pairs of rabbits can be produced from a single pair in a year if it is supposed: (*a*) that every month each pair begets a new pair which from the second month on becomes productive; and (*b*) that deaths do not occur? In this way we are led to the famous recurrent FIBONACCI series: 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$, in which each term after the second is the sum of the two preceding [147].

Among many other things the work contains also a proof of the well-known algebraic identity expressing the product of the sums of two squares as the sum of two squares: $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2$.

LEONARDO wrote two other important works: *Liber Quadratorum* [148] in 1220 and *Practica Geometriae* in 1225. The first of these is a brilliantly written, original, and able work on indeterminate analysis, stamping the author as the outstanding mathematician in the field from the time of DIOPHANTUS to the time of FERMAT over 400 years later. The great *Practica Geometriae* brings together a vast amount of material in geometry and trigonometry, and it would seem as if some works of the ancients now lost had been still available to LEONARDO. In particular this seems to have been true of EUCLID's work on the Division of Figures [72].

LEONARDO's great reputation led to a sort of mathematical tournament when JOHN of Palermo presented three problems which LEONARDO solved. The second of these was to find the solution of the equation $x^3+2x^2+10x=20$, the value for which, given without discussion in his *Flos*, correct to 10 places of decimals, has excited much wonder. Various surmises have been made as to his method of arriving at this result evidently based on Arabic methods [149].

Since we have already referred to NASÎR ED-DÎN [138] in the latter part of the thirteenth century, and ULUGH BEG [141] in the fifteenth century, we shall skip over the period of about 250 otherwise barren years and come to a German named **JOHANN MÜLLER**, born near Königsberg, Lower Franconia, and as a result known in the history of mathematics as **REGIOMONTANUS** [150]. He was the most influential and best-known German mathematician of the fifteenth century. We shall simply note that his work on trigonometry, written about 1464, but posthumously published in 1533, was the first European systematic

exposition of plane and spherical trigonometry; the only functions here introduced are sines and cosines [151]. The work had great influence in establishing trigonometry as a science independent of astronomy. The period of REGIOMONTANUS is also that of the Italian, LUCA PACIOLI, as well as that in which two mathematical works were printed, namely the first dated arithmetic [152] (Treviso, 1478), an anonymous commercial work, and the first edition of EUCLID'S *Elements*, a Latin translation by CAMPANUS (Venice, Ratdolt, 1482) [153]. PACIOLI [154] published two notable works, the first, usually referred to as *Sūma* (Venice, 1494), a great mathematical compilation drawn almost wholly from a variety of sources, and containing the first account of double-entry bookkeeping [155]; the second, *De divina proportione* (Venice, 1509), of special interest from the geometrical point of view, since the "divine proportion" here discussed was called "golden section" in the nineteenth century.

In the sixteenth century the chief Italian achievement was the solution of equations of the third and fourth degrees. The facts with regard to the general solution of the cubic equation seem to be as follows:

SCIPIONE DEL FERRO, a professor of mathematics at the University of Bologna, solved the equation $x^3+mx=n$ in 1515, imparting the result to his pupil ANTONIO FIOR, without publication. About 1535 NICOLO of Brescia who stammered badly because of an injury received as a child, and was therefore called TARTAGLIA, the stammerer, discovered the solution of the cubic $x^3+px^2=n$ as well as that of FERRO'S form. In a public contest provoked by FIOR who had been suspicious of TARTAGLIA'S achievements, TARTAGLIA triumphed completely [156]. Under a pledge of secrecy TARTAGLIA confided his method of solution to GIROLAMO CARDANO [157], a genius of great ability and the foremost Italian mathematician of his time, who taught mathematics and practised medicine in Milan. By far the most notable of the numerous works which CARDANO wrote was his *Ars Magna* published at Nürnberg, Germany, in 1545. This was the first great Latin treatise devoted solely to algebra. It was here that TARTAGLIA'S general solution of the cubic equation was first published without TARTAGLIA'S consent. Here, too, appeared the first solution of the general biquadratic equation [158]; this had been found by FERRARI, a pupil of CARDANO.

Already at this time the astronomers were feeling the need of trigonometric tables. After twelve years of incessant labor with computers, two works of extraordinary merit, and of value even to the present day, were finally prepared by RHETICUS or GEORG JOACHIM of Rhaetia, but not published until (1596, 1613) after his death (1576). One of the works was a ten-place table, with differences, of all six of the trigonometric functions, at interval 10″; the other was a fifteen-place table of natural sines, to every ten seconds of arc, with first, second, and third differences [159]. RHETICUS was the first: (i) to define trigonometric functions in connection with the ratios of sides of a right triangle; (ii) to employ the semiquadrantal arrangement of trigonometric tables. His table of secants was also the first table of the kind. RHETICUS was the leading mathematical astronomer in Teutonic countries in the sixteenth century. From 1539–

1542 he was a disciple of the Polish astronomer **Copernicus** [160] who after thirty-six years of labor had completed his great work *De Revolutionibus Orbium Coelestium* [161] whose publication in 1543 was due to importunities of Rheticus, like the *Principia* of Newton because of Halley. The Copernican Theory thus formulated had a great effect on thought of the time. We have seen that it had been put forward by **Aristarchus** about 1800 years earlier. A very influential sixteenth century English mathematician, **Robert Recorde** [162], was one of the first to bring the Copernican system to the attention of English readers.

The most influential of all the mathematicians in the Netherlands in the sixteenth century was **Simon Stevin** [163], famed in his time for his contributions to statics and hydrostatics. But he published the first compound interest tables [164] (1582), and in a tiny publication of 1585 he was the first to give a systematic explanation of decimal fractions. Here also he advocated the decimal division of the degree [164].

And finally we come to the greatest of all French mathematicians of the sixteenth century, **François Vieta** [165], a lawyer and Royal Commissioner who devoted most of his leisure to mathematics. His collected mathematical works form a considerable volume. He contributed extensively to the development of algebra and trigonometry [166]. He was among the first to employ letters to represent numbers in algebra, using vowels for the unknowns, and consonants for the knowns. He found the formula for $\cos n\phi$ in terms of $\cos \phi$ for any natural number n and wrote it down for values of n up to $n=9$; made an advance towards proving that a polynomial of the nth degree is made up of n linear factors; showed how to increase, decrease, multiply, or divide the roots of the equation $f(x)=0$, by k; gave the earliest evaluation of π as an infinite product [167]; applied algebra to geometry in such a way as to lay a foundation for analytic trigonometry; indicated powers more simply than his predecessors had done, using *A quadratum* or *A quad* for the square of the unknown, *A cubum* or *A cub* for its cube, *A quad quad* for its fourth power, and so on; and showed clearly the relation between the problems of the trisection of an angle and the solution of a cubic equation [65, 143].

In concluding our notes on mathematical developments during the 4700 years ending with the sixteenth century, we may draw attention to the fact that the first work on mathematics printed in the New World appeared at Mexico City in 1556, within 64 years of the discovery of America. It was the *Sumario Compendioso* by **Juan Diez** [168], and contained (*a*) tables intended to assist merchants in buying gold and silver, (*b*) an arithmetic suited to needs of apprentices in counting-houses, and (*c*) a few pages of algebraic problems, chiefly relating to quadratic equations.

HISTORY OF MATHEMATICS AFTER THE SIXTEENTH CENTURY

A. THE SEVENTEENTH CENTURY

The seventeenth century is especially outstanding in the history of mathematics. It saw FERMAT lay foundations for modern number theory, DESCARTES formulate algebraic geometry, PASCAL and DESARGUES open new fields for pure geometry, KEPLER discover laws of heavenly bodies, GALILEO GALILEI initiate experimental science, HUYGENS make notable contributions in the theory of probability and other fields, NEWTON create new worlds with calculus, curves, and physical observations, LEIBNIZ contribute notably in applications of the calculus and in mathematical notations, and NAPIER reveal a new method of computation. One cannot help being struck by the fact that in this century, with a single exception, all of these creators in mathematics were to be found in the north, where supremacy reigned to the end of the nineteenth century, in practically all fields.

While NAPIER, KEPLER, and GALILEO each lived many years in the sixteenth century, practically all of their important results to which we shall refer were obtained or announced in the seventeenth. It may be well to consider first the contributions of these men.

JOHN NAPIER [169] was a Scot, spent most of his life at Merchiston Castle near Edinburgh, and took an active part in the political and religious controversies of the day. As BALL expresses it [1]: "The business of his life was to show that the Pope was Antichrist, but his favorite amusement was the study of mathematics and science." In connection with the history of mathematics he is usually thought of only as the great inventor of logarithms; but the "rule of circular parts" (a mnemonic for readily reproducing the formulae used in solving right spherical triangles), the four formulae (known as "NAPIER'S analogies") for solving general spherical triangles, and the calculating rods [170], called "NAPIER'S Rods," for multiplying, dividing and taking of square roots, were also products of his genius. NAPIER seems to have had the idea of logarithms in mind as early as 1594, but it was not until 1614 that he published his researches in a "description of the admirable canon of logarithms" [171]. Besides explaining his logarithms, he gives a table of the logarithms of natural sines from 0 to 90° for each minute. The base of these logarithms is $q^{1/d}$, where $q=(1-10^{-7})$, $d=(1+\frac{1}{2}\cdot 10^{-7})10^{-7}$. Hence $q^{1/d}\approx e^{-1}(1-\frac{1}{2}\cdot 10^{-14})\approx e^{-1}$. Thus it is wholly incorrect to refer to NAPERIAN logarithms as natural logarithms [172].

While, from the time of ARCHIMEDES on to the time of NAPIER, there were numerous instances of the recognition of such a relation as $a^m\cdot a^n=a^{m+n}$, the law of exponents was in no sense a common idea, and it was not till later that the notion of a logarithm as an exponent became general. In NAPIER'S system the logarithm of 10^7 (not unity) was zero, although NAPIER recognized that a system with the logarithm of unity taken as zero was more desirable. It was effected by his friend, the English mathematician, **HENRY BRIGGS**, whose tables

of the logarithms of numbers from 1 to 20,000, and from 90,000 to 100,000, to 14 places [173], were published in 1624, seven years after NAPIER's death. The final part of a great new table to 20 places of decimals has recently appeared in England [174] in partial (though belated) celebration of the tercentenary of the discovery of logarithms. In Italy NAPIER's wonderful discovery was taken up enthusiastically by **BONAVENTURA CAVALIERI**, a pupil of GALILEO and professor of mathematics at the University of Bologna for the last 18 years of his life [175]; we shall presently have something to say about his work. In Germany KEPLER's zeal and reputation soon brought them into vogue there.

Let us now consider some of the contributions made by the second man of our group **JOHANN KEPLER** [176], mathematician, mystic, one of the founders of modern astronomy and an exceptional genius, outstanding as a brilliant calculator and patient investigator. In 1601 at Prague he became the assistant of the remarkable Danish astronomical observer **TYGE BRAHE** [176], who died a little later. The vast body of accurate observations of the planets which Kepler thus inherited led him in 1609 and 1619 to formulate the following laws:

I. All the planets move round the sun in elliptic orbits with the sun at one focus;

II. The line joining the planet to the sun sweeps out equal areas in equal intervals of time;

III. For all planets the square of the time of one complete revolution is proportional to the cube of the mean distance from the sun.

Purely as a thrilling intellectual experience, without any imaginable practical application, APOLLONIUS of Perga, and other Greeks, developed a marvellous body of knowledge with regard to conic sections. Then suddenly, 1800 years later, to a KEPLER this knowledge had most illuminating practical applications.

KEPLER's contributions to the infinitesimal calculus included finding volumes of surfaces of revolutions of conics about lines (93 different solids including the torus); deriving, in effect, the value of $\int \sin t\,dt$, and other integrals; and in solving maximum-minimum problems concerning cylinders, cones, and wine barrels. Of the four known star polyhedra KEPLER discovered two, the other two being found centuries later (1809) by LOUIS POINSOT. Star polygons and problems of filling a plane and space by regular figures were studied by KEPLER. He discovered how to determine whether a conic is a hyperbola, ellipse, parabola, or circle when given a vertex, the axis through it, and an arbitrary tangent with its point of contact. In one of his works, which consists rather in the enunciation of certain general principles, illustrated by a few cases, than in a systematic exposition, he laid down what has been called a Principle of Continuity, and gives as an example the statement that a parabola is at once the limiting case of an ellipse and of a hyperbola. He illustrated the same doctrine by reference to foci of conics; he explained also that parallel lines should be regarded as meeting at infinity. KEPLER contributed to the simplification of computations and published a volume of logarithms (1624–1625).

The last member of our group, **GALILEO GALILEI** [177], one of the most inter-

esting figures in the history of science, was the founder of the science of dynamics. His work in astronomy we shall not consider. A native of Pisa, he is said to have carried on experiments from the leaning tower there and to have arrived at the law that the distance of descent of a falling body was proportional to the square of the time, in accordance with the well-known law that $s=\frac{1}{2}gt^2$. GALILEO was the first to realize that, neglecting air resistance, the path of a projectile is a parabola. He speculated interestingly on laws involving momentum. He suggested that arches of bridges should be built in the form of cycloids [178], and by weighing pieces of paper he is believed to have found that the area of a cycloid is less than three times that of its generating circle. From 1609 on, he constructed telescopes.

Another mathematician who lived many years in the sixteenth century, although his outstanding publication appeared in the seventeenth century, was the Englishman THOMAS HARRIOT [179], who is of special interest to Americans since he was sent in 1585 to survey and map Virginia, now North Carolina. He died in 1621; one of his works published in 1631, *Artis Analyticae Praxis*, deals with algebraic equations of the first, second, third, and fourth degrees. It is more analytic than any algebra that preceded it, and marks an advance both in symbolism and notation, though negative and imaginary roots are rejected. It was potent in bringing analytic methods into general use. Harriot was the first to use the signs $>$ and $<$ to represent is greater than and is less than. When he denoted the unknown quantity by a, he represented a^2 by aa, a^3 by aaa, and so on. This is a distinct improvement on Vieta's notation. In unpublished manuscripts, attributed to HARRIOT, there is a table of binomial coefficients worked out in the form of a PASCAL triangle [180]. It was more than thirty years after HARRIOT's death that PASCAL is known to have used this triangle. MORLEY stated that the manuscripts contained a well-formed analytical geometry, but D. E. SMITH found that the manuscript in question was not in HARRIOT's handwriting.

HARRIOT's posthumous work appeared in the same year, 1631, as the first edition of the *Clavis Mathematicae of* WILLIAM OUGHTRED [181], a work on arithmetic and algebra. After NAPIER's *Descriptio* in 1614, this was the most influential mathematical publication in Great Britain, in the first half of the seventeenth century. It was one of the few books that contributed to laying the foundations of the mathematical knowledge of NEWTON as he was starting on his career. OUGHTRED exceptionally emphasized the use of mathematical symbols and, although he introduced more than a hundred of them, only three have come down to modern times. These are: our cross-sign for multiplication, the four-dot sign in proportion (::), and our sign for difference between ($\sim$). An OUGHTRED publication of 1618 contains [181a] (1) the first use of the sign $\times$; (2) the first abbreviations, or symbols, for the sine, tangent, cosine, and cotangent; (3) the first invention of the radix method of calculating logarithms; (4) the first table of natural logarithms. About 1622 OUGHTRED invented the slide rule [182], and his priority in the invention is unassailable. OUGHTRED was by

profession a minister of the gospel but his avocation was the teaching of mathematics and the writing of mathematical books. Among his pupils were JOHN WALLIS, and CHRISTOPHER, afterward SIR CHRISTOPHER, WREN. He was born in the last third of the sixteenth century and lived to be over ninety.

We may appropriately consider next the contributions of a member of the French group, **GIRARD DESARGUES**, born in the last decade of the sixteenth century. By profession an engineer and architect, his gratuitous lectures in Paris, when he was in his thirties, made a great impression on his contemporaries. Among his books was a treatise on how to teach children to sing well [**183**]. But it was his remarkable treatise on conics which stamps him as the most original contributor to pure geometry in the seventeenth century. DESARGUES commences with a statement of the doctrine of continuity as laid down by KEPLER and develops fundamental theorems on involution, homology, poles and polars, and perspective [**184**], such as arise to-day in our courses on synthetic projective geometry. This subject was not developed further, to any extent, until the early part of the nineteenth century.

When a mathematician hears the name PASCAL he usually thinks of another Frenchman, **BLAISE PASCAL** [**185**], author of the famous *Provincial Letters*, and not of his father **ÉTIENNE** whose name is connected with the quartic curve called PASCAL'S limaçon [**186**]. BLAISE showed phenomenal ability in mathematics at an early age, and DESCARTES could not at first credit that the manuscript on conics, which BLAISE wrote at the age of 16, had been written by him and not by his father. It was here that the famous PASCAL'S theorem occurs: If a hexagon be inscribed in a conic the points of intersection of opposite sides will be collinear [**187**]. In 1642, at the age of 19, he invented a computing machine which he later improved [**188**]. This, and one constructed by LEIBNIZ about 1694, were the first of their kind. PASCAL'S last work was on the cycloid. In correspondence with Fermat he laid down the principles of the theory of probability [**189**].

In connection with pure geometry in the seventeenth century it should now be noted that a work published at Amsterdam in 1672 by a Dane named **GEORG MOHR** [**190**] contained the elegant basis of a proof that all constructions with ruler and compasses could be carried through with compasses alone. Up to the recent republication of this work, it was generally thought that this theorem was first proved 125 years later by a capable Italian mathematician **LORENZO MASCHERONI**, and the term "MASCHERONI constructions" (which should much more correctly be "MOHR constructions") is current in mathematical literature.

The development of analytic geometry and of the infinitesimal calculus diverted attention from pure geometry. Let us now consider the work of the Frenchman, **RENÉ DESCARTES** [**191**], born in a family of wealth and culture, who greatly enriched the world by his writings in many fields. We are chiefly interested in his *La Geometrie* [**192**], a section of his large epoch-making work published at Amsterdam in 1637. The work has the title, *Discours de la Methode Pour bien conduire sa raison, & chercher la verité dans les sciences. Plus La Dioptrique. Les Meteores. Et La Geometrie. Qui sont des essais de cette Methode.* In the ADAM and TANNERY edition of *Œuvres de Descartes*, the whole fills 515 pages.

Of these *La Geometrie* occupies pp. 367–485, 511–514. The essence of plane analytic geometry is the study of loci by means of their equations. This was an important part of the investigation of conic sections by the Greeks. Even in the time of MENAECHMUS, the reduction of the problem of the duplication of the cube to finding two mean proportionals between two quantities was connected with parabolas and hyperbolas, and the equivalent in geometric form of their equations, as we now use them. APOLLONIUS of Perga established geometric relations equivalent to the equations of conics referred to a tangent and a diameter through its point of contact as axes. In the Greek scheme of variables a line segment corresponded to a first degree variable; the area of a rectangle, to the product of two such variables; the volume of a rectangular parallelopiped, to the product of three variables. Hence, after linear, plane, and solid magnitudes, the Greeks could go no further. On this DESCARTES made a tremendous advance by arithmetizing his geometry, and using excellent algebraic symbolism. The expression x^2 did not suggest to him an area but the fourth term in the proportion $1:x=x:x^2$. He laid off x on a certain line and then a length y at a fixed angle and sought points whose x's and y's satisfied certain conditions. He applied his method to the famous three- and four-line locus, and showed not only that this was a conic, but also that generalized forms of the problem, like that of PAPPUS, gave loci of higher degree. He showed that in the case of each of several curves mechanically constructed we could translate the mechanical process into algebraic language and so find an equation for the curve. The quartic ovals that bear his name [**193**] occur in this work. In the third book DESCARTES suggests that his method of geometric determination of a root may be extended to any degree. Here also he made use of what we now call "DESCARTES's rule of signs," distinguished between algebraic and transcendental curves, introduced the system of indices now in use, as a^4, and set the fashion of denoting variables by x, y, z, and constants by a, b, c.

DESCARTES was the first one to formulate the theorem, commonly attributed to EULER, on the relation between the number of faces, edges, and vertices of a convex polyhedron, $F+V=E+2$. He did not completely imagine the so-called folium of DESCARTES [**193**], $x^3+y^3=3axy$.

Another great French mathematical genius in the seventeenth century was **PIERRE DE FERMAT** [**194**], who was counselor for the local parliament at Toulouse, and who, except during 1641–53, devoted most of his leisure to mathematics after he was thirty years of age. He was the founder of the modern theory of numbers and a master in the field of DIOPHANTINE analysis [**195**]. Many theorems, of which he left no proof, have later been proved to be correct. Since it appears that practically no positive statement which he made has been shown to be incorrect, extraordinary celebrity has been developed in connection with the following theorem of which FERMAT stated that he had a proof: No integral values of x, y, z, can be found to satisfy the equation $x^n+y^n=z^n$ if n is an integer greater than 2. For, even to the present day, no mathematician has been able either to prove or to disprove the statement.

The numbers $F_n=2^{2^n}+1$ are often called FERMAT numbers because FERMAT

thought that F_n might possibly be a formula for giving an infinity of primes; F_n is prime for $n=0, 1, 2, 3, 4$, but it has since been shown that it was not prime for 13 other values of n. The numbers F_n became famous through the fact that about 1800 GAUSS discovered that when F_n is prime, it represents the number of sides of a regular polygon which can be constructed with ruler and compasses [196].

FERMAT was in possession of the general idea of finding maxima and minima; he also obtained the subtangents and areas of a number of curves, certain quadratures, and the center of mass of a paraboloid of revolution.

FERMAT gave an analytic treatment of the straight line and conic sections, and he and PASCAL were the founders of the theory of probability, but they published nothing on this subject.

In reviewing NAPIER'S work we found **CAVALIERI** enthusiastically advocating the use of logarithms. CAVALIERI'S chief contribution to mathematics was his principle of indivisibles discussed somewhat by ARISTOTLE, and also receiving the attention of GALILEO, TORRICELLI, and ROBERVAL [197]. It occupies a place intermediate between the method of exhaustion of the Greeks, and the calculus methods of NEWTON and LEIBNIZ. Each indivisible is capable of generating the next higher continuum by motion; a moving point generates a line, a moving line a plane, a moving plane a solid. Though lacking in scientific foundation it was a sort of integral calculus which yielded solutions of difficult problems. A theorem called CAVALIERI'S theorem [198] has recently come into use in America in the teaching of elementary geometry, as a means of unifying the ideas that lie at the basis of the mensuration of solids. The substance of this theorem is as follows: If two solids have equivalent bases, and if sections parallel to the bases and equally distant from them in the two solids are also equivalent, then the solids are equivalent.

On turning once more to the north the most eminent mathematician in Scotland in the seventeenth century after NAPIER was **JAMES GREGORY** [199], for a few years professor of mathematics at the Universities of St. Andrews and Edinburgh. In 1671 he was an independent discoverer of the binomial theorem in its most general form, of the interpolation formula (usually referred to as GREGORY-NEWTON), and of expansions including those for $\tan x$, $\sec^{-1} x$, and $\tan^{-1} x$; for this last

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots,$$

from which when $x=1$, $\pi/4$ may be expressed (not noted by GREGORY) as a series (known to LEIBNIZ in 1674). He evaluated

$$u = \int_0^{\alpha} \sec\theta d\theta = \ln(\sec\alpha - \tan\alpha) = \ln\tan(45^\circ + \tfrac{1}{2}\alpha),$$

thus settling an Edward Wright problem [200], discovered (1671–72) TAYLOR'S (1735) theorem, and, inspired by FERMAT, solved DIOPHANTINE problems.

From GREGORY'S series for $\tan^{-1} x$, JOHN MACHIN, an eighteenth century professor of astronomy [201], derived the relation (1706)

$$\frac{1}{4}\pi = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239},$$

which was used to calculate π to 707 decimal places (1874). In 1947 (with another formula) it was shown by another English computer, who calculated π to 808 decimal places, that the earlier extended computation was correct to only 526 decimal places.

On the continent we find for our story much of interest in the work of CHRISTIAAN HUYGENS, eminent as physicist, astronomer, and mathematician [202], the greatest scientist that Holland has ever produced. Most of his discoveries were made with the aid of ancient Greek geometry, for which, like his friend NEWTON, he always showed partiality; but at times he called to his aid ideas suggested by DESCARTES, CAVALIERI, and FERMAT. NEWTON and HUYGENS were kindred spirits and always had the greatest admiration for each other.

ARCHIMEDES secured his approximation to the value of π correct to two places of decimals by considering inscribed and circumscribed polygons of $96 = 3.2^5$ sides. Using the same method VIETA calculated the value correctly to nine places of decimals by means of polygons of $393\,216 = 3.2^{17}$ sides, and Ludolf van Ceulen to 20 places by means of polygons of $32\,212\,254\,720 = 60.2^{29}$ sides; after spending most of his life in calculation he used this same method to determine the result to 35 places of decimals. As a result of a series of geometric propositions given by HUYGENS in his remarkable work on the magnitude of a circle, the ARCHIMEDES' value may be obtained from an inscribed hexagon, and a value correct to 9 places from a regular polygon of only sixty sides [203].

HUYGENS wrote also the first formal treatise on probability, which was the best until superseded by more elaborate works of JAMES I BERNOULLI and DEMOIVRE. To the theory of curves he added the important theory of evolutes and showed in particular that the evolute of a cycloid is an equal cycloid, which led to his construction of the isochronal pendulum. Stimulated by the astronomer's need, he invented the pendulum to regulate the movement of clocks. HUYGENS showed also that the cycloid has the property that when the accelerating force is gravity a particle moving along a vertical curve will arrive in the same time at a given point wherever the initial point is taken on the curve. HUYGENS solved the problem of determining the equation of the catenary curve formed by a perfectly flexible chain suspended by its ends. He found also many other results concerning curves, some of them generalizations of particular cases considered by FERMAT, and dealt with many interesting problems of pure geometry.

In England of the seventeenth century the work of HARRIOT and of OUGHTRED has been considered; but a much abler mathematician, one of the most

original of his day was JOHN WALLIS [204], for 54 years professor of geometry at Oxford University. He was a man of great erudition, and wrote works in a dozen different fields. His great work on *Algebra* (not without a blemish) contains the first record of an effort to represent an imaginary number graphically; while different from the method now used, it is complete and consistent in itself. In discussing fourth dimension he refers to it as a "Monster in Nature, and less possible than a *Chimaera* or *Centaure.*" He used CAVALIERI's method of indivisibles to find the areas of curves $y=x^m$ when m is a positive integer or a fraction. He failed to get the approximate quadrature of the circle $y=(1-x^2)^{1/2}$ by series, since he did not know the binomial theorem for a fractional power; but by an original method of interpolation he expressed π in the form of an infinite product,

$$\frac{4}{\pi}=1\times\frac{9}{8}\times\frac{25}{24}\times\frac{49}{48}\times\frac{81}{80}\cdots;$$

thus bringing us to the beginning of the second period in the history of the problem of squaring the circle [67], the first having ended with discussions of polygon perimeters.

But the greatest scientist of any century was ISAAC NEWTON [205–208], who was born December 25, 1642, o.s., or January 5, 1643, n.s., and died in 1727, aged 84. Already at Cambridge University, at the age of 22, in the year that he took his B.A. degree, he had invented the method of fluxions [209] or calculus, and discovered the binomial theorem [209]. In the following year he made applications of the calculus to tangents, curvature, concavity and convexity, maxima and minima, and other things, some of which were new and of importance. It was about this time also that he made interesting discoveries in optics and formulated, but did not announce, the law of universal gravitation. Referring to these years later in life NEWTON wrote, "in those days I was in the prime of my age for invention, and minded Mathematicks and philosophy more than at any time since."

His former teacher ISAAC BARROW [210], fully recognizing the transcendent ability of NEWTON, resigned his chair (although only thirty-nine years old) to make way for NEWTON, then only twenty-six years of age, to be appointed in his place as professor of mathematics at Cambridge. At the age of forty-seven NEWTON represented the University in Parliament and by the time he was fifty his scientific creations were practically ended; but in later years he received many honors and a number of his earlier writings or lectures were published.

His greatest work was indisputably the "Mathematical Principles of Natural Philosophy" usually known by a single word of the Latin title, *Principia* [211], published in 1687. This work (1) treats of the motion of particles or bodies in free space either in known orbits, or under the action of known forces or under mutual attractions; (2) treats of motion in a resisting medium, and of hydrostatics and hydrodynamics with special applications to waves, tides, and acoustics; and (3) applies theorems obtained to the chief phenomena of the solar system and determines the masses and distances of the planets. The motions of

the moon, the various inequalities therein, and the theory of tides are worked out in detail. The great principle underlying the whole work is that of universal gravitation characterized as "indisputably and incomparably the greatest scientific discovery ever made." Possibly because he regarded the presentation as more convincing for students of his time, NEWTON achieved the stupendous task of translating his arguments into the language of Greek geometry even though many of his results were first worked out by methods of the calculus. In the *Principia* are a great many results concerning well-known curves, such as epi- and hypocycloids, and many beautiful results in geometry, such as the following: (1) If a point P moving along a given straight line is joined to two fixed points O and O', and if lines OQ and $O'Q$ make fixed angles with OP, $O'P$, then the locus of Q is a conic. (2) The locus of the centers of conics tangent to the sides of a quadrilateral is the line ("NEWTON's line") [212] through the centers of its diagonals (lemma 25, cor. 3).

NEWTON studied curves of the third order [213] with a view, apparently, to illustrating the application of analytic geometry to curves other than conics. He enumerated 72 of the possible 78 forms which a cubic may assume, in his classification; and in the course of the work he states the remarkable theorem that just as the shadow of a circle cast by a luminous point on a plane gives rise to all the conics, so the shadow of the curves represented by the equation $y^2=ax^3+bx^2+cx+d$ gives rise to all cubics. This remained an unsolved puzzle till a proof was discovered in 1731.

In another monograph on the quadrature and rectification of curves by means of infinite series, NEWTON indicates at one point the importance of determining whether the series are convergent—an observation far in advance of his time—but there is no record that he knew a general test for the purpose; and in fact it was not until GAUSS and CAUCHY took up the question, early in the 19th century, that the necessity for such limitation was commonly recognized. In this monograph $\arcsin x$ is expanded in a series, and the process of reversion of series is employed.

In his *Universal Arithmetic*, containing the substance of lectures [214], DESCARTES's rule of signs is extended to give limits to the number of imaginary roots of an equation, and a method is given for finding the approximate values of the roots of a numerical equation [215], a method which applies equally to algebraic or transcendental equations. He enunciated also the theorem, known by his name, for finding the sum of the nth powers of the roots of an equation, and laid the foundation of the theory of symmetric functions of the roots of an equation.

In connection with various challenges among mathematicians, NEWTON was never beaten; in one from LEIBNIZ he solved the problem of finding the orthogonal trajectories of a family of curves. One of the most interesting eulogies on NEWTON's work is that attributed to LEIBNIZ who, upon being asked by the Queen of Prussia what he thought of NEWTON, answered: "Taking mathematicians from the beginning of the world to the time when NEWTON lived, what he had done was much the better half" [216].

LEIBNIZ [217], the only pure mathematician of the first class produced by Germany during the seventeenth century, was equally notable in the field of philosophy. The last years of his life were embittered by a controversy as to whether he had discovered the calculus independently of NEWTON's previous investigations, or had derived the fundamental idea from NEWTON. There is now no doubt that LEIBNIZ developed his calculus quite independently, and that he and NEWTON are each entitled to credit for their respective discoveries. The two lines of approach were radically different, although the respective theories accomplished results that were practically identical. LEIBNIZ was original in much that he did. He made the CAVALIERI method scientific. He introduced the idea of fractional differentiation [218]. He laid the foundation for the theory of envelopes and defined the osculating circle, showing its importance in the study of curves.

The history of determinants [219] begins with LEIBNIZ in connection with the eliminant of linear equations. He gave a generalization of NEWTON's binomial expansion rule in his expansion of any polynomial $s+x+y+z+\cdots$ to any arbitrary power r. We have already referred to the calculating machine [220] he constructed.

LEIBNIZ made important contributions to the notation of mathematics [221]. Not only is our notation of the differential and integral calculus due to him but also our signs, in geometry, for similar ($\sim$), and for equal and similar, or congruent, ($\simeq$).

And finally, among the names of mathematicians of the last half of this century we shall briefly note results found by JAMES I BERNOULLI of Switzerland, a member of a remarkable family in which 8 members distinguished themselves in mathematics [222]. He was professor of mathematics at the University of Basel, and was interested in the fields of astronomy, mathematics, and physics. He was the first to solve the problem of isochronous curves proposed by LEIBNIZ, that is, to find the curve along which a body falls with uniform vertical velocity; in his solution of this problem (1690) we meet with the word integral for the first time. HUYGENS also solved this problem and found it to be the semicubical parabola [69] $x^3=ay^2$.

He worked on the problem: To find the curvature a lamina should have in order to be straightened out horizontally by a weight suspended at the end. He noted the relation $Rs=a^2$, where R is the radius of curvature at a distance s along the lamina from the point of suspension, and gave a construction for points of the curve. But it was EULER in the next century, who first really visualized the curve as a double spiral [223] with asymptotic points, later called the clothoïde [69]; the equation of the spiral is:

$$x=\int_0^s \sin(\tfrac{1}{2}t^2/a^2)\,dt, \qquad y=\int_0^s \cos(\tfrac{1}{2}t^2/a^2)\,dt.$$

This is the spiral which comes up in connection with FRESNEL's discoveries of

the diffraction of light, and in railway easement curves [224]. It has a number of interesting properties.

BERNOULLI did much work on the logarithmic spiral [225] and found for example that (1) the evolute of a logarithmic spiral is another equal logarithmic spiral, (2) the pedal of a logarithmic spiral with respect to its pole is an equal logarithmic spiral; (3) the caustic by reflection and refraction of a logarithmic spiral for rays emanating from the pole as a luminous point is an equal spiral. Such perpetual renascence delighted BERNOULLI and it was, in accordance with his directions, recorded on his tomb with the inscription: *Eadem mutata resurgo* —"I arise the same though changed."

BERNOULLI wrote also a posthumously published (1713) work entitled *Ars Conjectandi* which established principles of the calculus of probability [226] in more elaborate form than HUYGENS had done. It is here that we meet certain numbers, known in a very extensive later literature as "BERNOULLI numbers." In 1913 the Academy of Sciences at Leningrad celebrated the bicentenary of the law of large numbers, here published.

Thus closes our sketch of the seventeenth century during which such great advances were made in fields of analytic geometry, calculus and its applications, mechanics and dynamics, geometry, theory of probabilities and the theory of numbers.

B. THE EIGHTEENTH CENTURY

The early part of this century saw the beginnings of Analytic Geometry of Three Dimensions [227]. In 1679 PHILIPPE DE LA HIRE solved a simple problem; but by 1700 ANTOINE PARENT had gone much farther and before 1750 ALEXIS CLAIRAUT and JAKOB HERMANN were discussing problems in analytic geometry of curves and surfaces.

The first half of the century saw also the chief publication of one of the forerunners of noneuclidean Geometry, **GIROLAMO SACCHERI** [140, 228], Jesuit teacher of theology, logic, metaphysics, and mathematics, at Turin, Pavia, and Milan.

The principal mathematicians flourishing in this century were, DEMOIVRE, TAYLOR, MACLAURIN, EULER, D'ALEMBERT, LAGRANGE, LAMBERT and MONGE.

Since **ABRAHAM DEMOIVRE** [229], although born in France, spent 66 years of his life in England, he is properly considered a member of the English school. He was an intimate friend of NEWTON. Apart from numerous memoirs he published a work called *Annuities upon Lives*, of which there were seven editions, a work of importance in the history of actuarial mathematics. His very valuable *Doctrine of Chances* passed through three editions and contained much new material connected with the theory of probability [230]; indeed, he ranks with LAPLACE in making the most important contributions to the subject up to the end of the eighteenth century. He gave the first treatment of the probability integral, and essentially of the normal curve. He formulated and used a theorem improperly called STIRLING'S theorem [231].

In his *Miscellanea Analytica* [230], 1730, which brought about his election as a Fellow in the Berlin Academy, there is important material in connection with recurrent series, analytic trigonometry involving imaginaries, and problems of chance, and analysis is applied to problems of astronomy. From one part of this work it is clear that DEMOIVRE was thoroughly familiar with the formula now connected with his name [232]:

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$

where n is an integer. But the result is not explicitly stated in any of DEMOIVRE's writings; a form equivalent to this was, however, given in 1748 by EULER [232]. In this work DEMOIVRE studied also the factors of $x^{2n} - 2x^n \cos n\theta + 1 [= y]$. If O is the center of a unit circle circumscribing a regular polygon of n sides $A_1A_2 \cdots A_n$, if $OP = x$, and if $\angle POA_1 = \theta$, then $\overline{PA_1}^2 \cdot \overline{PA_2}^2 \cdot \overline{PA_3}^2 \cdots \overline{PA_n}^2 = y$, called "DEMOIVRE's property of the circle."

The name of the Englishman **BROOK TAYLOR** [233] is familiar to every student of the calculus, through the theorem of TAYLOR's expansion of $f(x+h)$, given in his *Methodus Incrementorum directa et inversa*. (This is the first treatise dealing with finite differences; it was published in 1715.) The convergency of the expansion is not considered.

In this work TAYLOR solved the following problem which he believed to be new: "To find the number of vibrations that a string will make in a certain time, having given its length, its weight, and the weight that stretches it." In discussing the form of the vibrating string, his suppositions regarding initial conditions, including that it vibrated as a whole, led to a differential equation whose integral gave a sine curve. Thus started a discussion which was to culminate a century later in the work of Fourier [234].

We shall next consider **COLIN MACLAURIN** [235], a Scot, and one of the ablest mathematicians of the eighteenth century. His name is associated with TAYLOR in the expansion of $f(x+h)$, when $h = 0$. This was given in his *Treatise of Fluxions* in 1742, but the same thing appeared twenty-five years earlier, in a publication by STIRLING. MACLAURIN's *Treatise* is especially valuable for the solutions it contains of numerous problems in geometry, statics, the theory of attractions, and astronomy. He followed NEWTON in using geometric methods. By avoiding the use of analysis and of the infinitesimal calculus he helped to keep alive methods not abandoned in England till about 1820. In 1740 MACLAURIN shared with EULER and D. BERNOULLI a prize of the French Academy of Sciences for his memoir on the tides, winning from LAGRANGE the highest praise [236].

When only 21 years of age MACLAURIN wrote a work called *Geometria Organica* which was inspired by NEWTON's work on curves of the third order, and by the organic description of a conic which we quoted from NEWTON's *Principia*. He attempted to generalize this organic description of a conic so as to obtain curves of all possible degrees by a mechanical description. He discusses also various pedal curves [237]. In this way he was led to such familiar curves as the cissoid, strophoid, cardioid, limaçon, and lemniscate, and many

others [237] which have since become notable in connection with the study of special curves.

We have already more than once referred to LEONARD EULER [238], the great Swiss mathematical genius, born in 1707 at Basel, the home of the BERNOULLIS. The project of erecting an Academy at Petersburg, which had been formed by PETER THE GREAT, led to EULER being invited in 1727 to the chair of mathematics there. After fourteen years he accepted the invitation of FREDERICK THE GREAT [238a] to go to Berlin to direct the Prussian Academy which was then rising in fame; he continued there for twenty-five years, until 1766, when he returned once more to Petersburg, where he remained till his death in 1783 at the age of 76.

Few writers in mathematics ever contributed so extensively or so fruitfully. In a sense he was the creator of modern mathematical expression. He revised almost all the branches of mathematics then known, filling up details, adding proofs and new results, and arranging the whole in a consistent form. More than 700 memoirs and papers came from his hand, besides many treatises, half of them being published after he was totally blind. He wrote on philosophy, astronomy, physics, geography and agriculture; and on mathematics in the following fields, among others: arithmetic, theory of numbers, Diophantine equations, algebraic equations, series, continued fractions, calculus of probabilities [239], analysis, definite and indefinite integrals, elliptic functions and integrals [240], differential equations, calculus of variations, elementary geometry, analytic geometry, differential geometry, mathematical recreations, music [241], mechanics, ballistics, and the motion of ships.

Certain of our ordinary notations [242] originated with EULER,

$f(x)$ for function symbol,
e for $2.71828\cdots$,
s for half the sum of the sides of a triangle,
a, b, c for the sides of a triangle ABC,
$\sum$ for summation sign,
i for $\sqrt{(-1)}$.

From the formula $e^{ix}=\cos x+i\sin x$, for $x=\pi$, $e^{i\pi}+1=0$. This relation was to be the basis of a proof that the problem of squaring the circle was impossible [243]. It should be noted that thirty years before EULER published his result ROGER COTES gave the equivalent in words of the relation:

$$ix = \log(\cos x + i \sin x).$$

EULER showed that $\frac{1}{2}\pi=-i\log i$, and that $(\log i)/i$, and i^i had an infinite number of values [244]. He gave the formula,

$$\pi = 20\tan^{-1}\frac{1}{7} + 8\tan^{-1}\frac{3}{79},$$

and from it calculated π to 20 places of decimals.

EULER showed that the center of the circumscribed circle of a triangle, its center of gravity, and the point of meeting of the altitudes are collinear [245]; hence the name *Euler's Line*. It was later shown that the center of the nine-point circle is also on this line.

EULER gave a new method of solving the general equation of the fourth degree. He introduced the idea of an indicatrix for studying the variations of the radii of curvature of normal sections of a surface at a point. He worked on the problem of a vibrating string [241] initiated by BROOK TAYLOR, and later carried on by Lagrange, preparatory to the important work of FOURIER [246] in the early part of the 19th century. It was in his work on the calculus of variations that Euler considered the problem of JAMES I BERNOULLI for a coiled lamina [238], to which reference has been made. Beta and gamma [247] functions originated with EULER, as well as the EULER constant [248],

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.5772156649\cdots .$$

When EULER left Berlin in 1766, he was succeeded by JOSEPH LOUIS LAGRANGE [249], an Italian by birth, a German by adoption and a Frenchman by choice, one of the great mathematicians of all time. He was then 30 years old. In sending the invitation FREDERICK THE GREAT wrote that the greatest king in Europe wanted the greatest mathematician of Europe at his court. LAGRANGE remained there till the death of FREDERICK 20 years later. Many of his great memoirs, as well as the composition of his monumental work on *Analytic Mechanics* (containing his well-known general equations of motion of a dynamical system), date from this period. During the remaining 25 years of his life he was in Paris, a professor at the newly established École Normale Supérieure, and École Polytechnique. These two great schools occupy a most important place in the history of mathematics. Most of the greatest mathematicians of France have been professors in them, and practically all have there been trained. Towards the close of the eighteenth century appeared two more of LAGRANGE's great works, the *Resolution of Numerical Equations of Every Degree*, and the *Theory of Analytic Functions Containing the Principles of the Differential Calculus*. In the first of these he attempted a proof (see later, under GAUSS), that every algebraic equation has a root. His work in the field of Calculus of Variations was important, and his name is linked with that of EULER in founding the subject. His contributions in the field of Differential Equations were notable. Like DIOPHANTUS and FERMAT, he had a special genius for the theory of numbers. As a whole his work had a very profound influence on later mathematical research.

Let us now turn back to Germany to consider contributions made by JOHANN HEINRICH LAMBERT [250], who enjoyed association with EULER and LAGRANGE in Berlin. He was a many-sided scholar and wrote on a great variety of subjects.

His *Freye Perspective* of 1774 contained numerous geometric problems of interest [250].

In a monograph of 1761, on the determination of a comet's orbit, he gave incidentally the results that a circle about a triangle circumscribing a parabola goes through the focus; and hence if four tangents are given the focus is determined. From the rediscovery of this parabola property, nearly forty years later, WILLIAM WALLACE [251] deduced his theorem: The feet of the perpendiculars on the sides of a triangle, from any point on its circumscribed circle, are collinear.

In 1767 LAMBERT had proved with most minute rigor [252] that π, e^x, and $\tan x$ were irrational. (In 1737 EULER had shown that e and e^2 were irrational.) This result for π was the beginning of the third period in the history of the problem of the squaring of the circle [67]. In 1844 and 1851 a prominent French mathematician named LIOUVILLE [253] proved the existence of transcendental numbers, that is of numbers which are not roots of algebraic equations with integral coefficients. In 1873 a great French mathematician named HERMITE [254] proved that e was transcendental. As a result, and by means of EULER's relation $e^{i\pi}+1=0$, it was shown, in 1882, that π was transcendental, and hence that the problem of squaring the circle was impossible of solution with ruler and compasses. This last result was found by LINDEMANN, a German.

To LAMBERT we owe the systematic development of the theory of hyperbolic functions, introduced by VINCENZO RICCATI a few years earlier [255]. LAMBERT's name is also associated with important projections used in map making [256].

Three other French mathematicians of prominence, MONGE, LAPLACE, and LEGENDRE, were born about the middle of the eighteenth century, but the culminating period of the most important work of the two latter was in the nineteenth century. For the final name in the eighteenth century, we shall, then, consider that of GASPARD MONGE [257], a professor of mathematics in the École Polytechnique of Paris. He was the inventor of descriptive geometry, and his treatise on this subject, which appeared first in 1794, went through a number of editions. There were five editions of his *Applications of Analysis to Geometry* which first appeared in 1795; it was an important early work on the differential geometry of surfaces.

Our survey has suggested that while the eighteenth century saw the further development of such subjects as trigonometry, analytic geometry, analysis, theory of numbers, differential equations, probabilities, and analytic mechanics, it saw also the creation of new subjects such as actuarial science, calculus of variations, finite differences, gamma and elliptic functions, descriptive geometry, and differential geometry [258].

C. THE NINETEENTH CENTURY AND LATER

MORITZ CANTOR's great history of mathematics to the end of the eighteenth century contained four large volumes [12] with an average of 988 pages to the

volume. It has been estimated that if the history of mathematics in the nineteenth century were written with the same detail it would require fourteen or fifteen more similarly sized volumes [259]. As indicated in our introductory remarks we shall, in conclusion, touch very briefly on only a very few topics in the enormous mass of available material.

We saw that LAPLACE and LEGENDRE were two great French mathematicians who were contemporaries of LAGRANGE, but whose principal work was published in the nineteenth century. Let us now consider them. **PIERRE SIMON LAPLACE** [260], born in obscurity, became titled through political activities. He participated in the organization of the École Normale Supérieure and École Polytechnique. His researches were carried on chiefly in the fields of astronomy, celestial mechanics, probabilities [261], calculus, differential equations, and geodesy. In 1796 he published a non-mathematical popular treatise on astronomy entitled, *Exposition of the system of the world.* It was followed by his monumental mathematical *Celestial mechanics*, embracing all the discoveries in this field, of NEWTON, EULER, LAGRANGE, and others who preceded him, as well as his own. This work stamped him at the time as an unrivalled master in the field. (Four volumes were translated into English with elaborate commentary by NATHANIEL BOWDITCH, the New England author of the *New American Practical Navigator* [262], which has passed through more than 65 editions since the first in 1799.) In 1812–1814 LAPLACE published both popular and mathematical treatises on probability. He was characteristically neglectful of crediting predecessors with their discoveries. It may well be, then, that LAPLACE knew that DEMOIVRE had treated the probability integral, as well as what is essentially the normal curve, long before him [231].

The notion of a potential function introduced into analysis by Lagrange was much used by LAPLACE who showed that if V is such a function it satisfies the partial differential equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

called LAPLACE'S equation.

NEWTON died in 1727, and LAPLACE in 1827. How vast the mathematical advances in that century!

ADRIEN MARIE LEGENDRE [263] is chiefly known in higher mathematics for important works on the theory of numbers (1798, third ed., 2 v., 1830), on elliptic functions and integrals [264] including great tables, the result of 30 years of meditation (3 v., 1825–1832), and on exercises in the integral calculus (3 v., 1811–1817). Most of these works deal with topics in higher mathematics the discussion of which is beyond the scope of this sketch. LEGENDRE (1752–1833) was the first one to publish the method of least squares (1805) [265]. LEGENDRE wrote also a work on the elements of geometry and trigonometry, first published in 1794, which has gone through many editions [266]. This work

is based on EUCLID's Elements, but rearrangements and simplifications are introduced without lessening the rigor of the ancient methods of treatment. In successive editions LEGENDRE tried to prove the fifth postulate of EUCLID, and every student of the foundations of geometry takes account of results thus found [266]. The work contains also a proof that both π and π^2 are irrational, a proof of the first result having been already given by LAMBERT [252].

A geometry which is built up without assuming EUCLID's fifth postulate as true is said to be noneuclidean. Such a geometry was worked out almost simultaneously by a Russian and a Hungarian. The Russian, NICOLAI LOBACHEVSKY [267], studied at the University of Kazan and became a professor of mathematics there in 1814 at the age of twenty-one. In 1826 he made known, through his lectures, his conception of such a geometry which should not depend upon the Euclidean postulate of parallel lines. These ideas were published in 1829 and later. The Hungarian, JÁNOS BOLYAI [268], whose results were worked out in 1825–26, published them in 1832. The results thus presented were, later in the century, developed into a body of theory of high importance in mathematics [269].

The fundamental ideas of noneuclidean geometry were also at this time familiar to CARL FRIEDRICH GAUSS [270], who spent most of his life at Brunswick and Göttingen, Germany. As a child of three he detected an error in his father's calculation of the wages of an employee; he became the greatest mathematician that Germany, and one of the greatest mathematicians that the world, has ever known. He was born in 1777 and was led to take up the study of mathematics by his discovery in 1796 that it was possible to construct a regular polygon of 17 sides by means of a ruler and compasses. He was later able to show that when the FERMAT number $F_n = 2^{2^n}+1$ is prime, it represents the number of sides of a regular polygon constructible with ruler and compasses [271]. He did not prove, however, what is a fact, namely that these are the *only* polygons, the number of whose sides is prime, which are so constructible. Only four such polygons are known at present. This discussion occupied a very small part of GAUSS's greatest single work, *Disquisitiones Arithmeticae*, which deals with matters of fundamental importance in the theory of numbers. His memoirs of 1825–27 on general investigations of curved surfaces, fundamental in differential geometry, were translated into English, edited with a bibliography showing resulting developments, and published by Princeton University (1902).

GAUSS worked and created in many other fields, analytic functions of a complex variable, astronomy [272], geodesy, electromagnetism, least squares, and almost every leading branch of mathematics. He gave the first wholly satisfactory proof that every algebraic equation of degree m has exactly m roots [270] (which GAUSS calls the Fundamental Theorem of algebra), in his doctoral dissertation written when he was twenty years of age. He here shows defects in the earlier consideration of this question by D'ALEMBERT [273] (1746), EULER (1749), FONCENEX (1759), and LAGRANGE (1772). GAUSS gave three other proofs of this theorem in 1816 and 1850. He died in 1855 aged 78.

GAUSS was a notable instance of a youthful arithmetical prodigy who later achieved greatness in mathematics. The most extraordinary mental calculating prodigy who ever lived was ZACHARIAS DASE [274] of Hamburg who died in 1861, aged 37. He multiplied two 8-figure numbers in 54 seconds, two 20-figure numbers in 6 minutes, two 40-figure numbers in 40 minutes and two 100-figure numbers in $8\frac{3}{4}$ hours, and found the square root of a 100-figure number in 52 minutes. GAUSS tried to turn his abilities to some useful purpose, and as a result we have DASE's seven-place tables of the natural logarithms of numbers from 1 to 10,500 (1850) and his factor tables of all numbers in the seventh, eighth, and ninth (with assistance) millions (1862–1865).

In considering the mathematics of the seventeenth century we referred to the notable development of synthetic geometry by PASCAL and DESARGUES. There was practically no extension of their work until the early part of the nineteenth century when enormous strides were made by the Frenchman VICTOR PONCELET [275] and the Swiss JAKOB STEINER [276], who spent most of his life in Berlin. It was PONCELET who first indicated that every construction with ruler and compasses is also possible: (1) with ruler alone, if a single circle and its center are given in the plane; and (2) with an angle of given opening, α, alone [277]. Very early in the nineteenth century the principle of duality was formulated, and BRIANCHON [278], a student in the École Polytechnique, enunciated the dual of PASCAL's theorem, namely, The three diagonals of a hexagon circumscribing a conic are concurrent.

We have referred to the proof of the impossibility of the problem of squaring the circle. The other two famous problems of the ancients lead to cubic equations. In 1837 these problems also were shown to be impossible of solution with ruler and compasses, when it was proved [279] that the corresponding cubic equations were irreducible in Euclidean domain.

The transformation of figures in the plane by circular inversion is of importance in both pure and applied mathematics. This seems to have been first conceived [279a] by STEINER in 1824, and by the Belgian astronomer and statistician ADOLPHE QUETELET in the following year. German, Italian and Irish expounders of the method soon followed.

An outstanding geometrical problem in the latter part of the nineteenth century was to discover a linkage for drawing a straight line. A beautiful arrangement of seven jointed rods was finally devised in 1864 by a Frenchman named PEAUCELLIER [280]. The construction of this was based upon the principle that the inverse of a circle through the center of inversion is a straight line. The same principle was used later in a new straight-line linkage containing only five links. Many linkages were found [281] for constructing special curves, such as conics, cardioids, lemniscates, and cissoids. It has been proved that a linkage is possible for tracing every algebraic curve, but that it is not possible to construct a linkage for tracing any transcendental curve. Linkages have also been used for solving algebraic equations.

We have seen that the solutions of the general equations of the third and

fourth degrees were found in the sixteenth century by TARTAGLIA and FERRARI. Nearly three hundred years were to elapse before it was shown that a similar solution by radicals of the general equation of the fifth degree was not possible. This was proved in 1799 by an Italian physician **PAOLO RUFFINI** [282], and in a pamphlet published in 1824 by the Norwegian genius **NIELS HENRIK ABEL** [264]. We have already referred [264] to ABEL'S priority in connection with elliptic functions.

Of methods for approximating to the real roots of numerical equations we noted that of NEWTON by successive substitutions in derived equations. This dates from 1669 although it was first published in WALLIS'S *Algebra* in 1685. It is the method which to this day has held undisputed sway in France. The so-called "HORNER'S method" in a paper read by **WILLIAM G. HORNER** [283] before the Royal Society of London in 1819, and published in the same year, was really given fifteen years earlier by the Italian PAOLO RUFFINI. It soon became widely used in England, and later in the United States, and to a less degree in Germany, Austria, and Italy. A valuable work on numerical equations by FOURIER was published in 1831, after his death [246]; but for nearly twenty-five years before this he had taught his students at the École Polytechnique the theorem now associated with his name, concerning the limit to the number of real roots in any given interval. In 1829 **CHARLES STURM** [284] stated a theorem for determining the exact number of roots in any interval; this was merely a by-product of his extensive investigations of linear difference equations of the second order, although it was suggested by a close study of the manuscript of FOURIER'S work of 1831. Considerable work has been done also in deriving the roots of numerical equations by means of infinite series [246].

The problem of determining the imaginary roots of numerical equations has been considered by many writers. LAGRANGE'S work contained a valid method. STURM'S theorem gave a method for determining the number of such roots, but not their location. This was revealed by a general theorem [285] of a great French mathematician **AUGUSTIN LOUIS CAUCHY** [286] (1789–1857), giving the number of roots, real or complex, which lie within a given contour (1831).

A few comments in connection with the convergency of infinite series, so necessary for consideration by every mathematician, are now appropriate. We noted that this question was discussed by NEWTON in a particular case. But the first important and strictly rigorous investigation of infinite series (compare LAMBERT [252]) was made by GAUSS in 1812 in connection with hypergeometric series. In 1821, however, CAUCHY gave a discussion and tests of general application, such as are now constantly in use. A paper of ABEL on power series, published in 1826, is also of importance in this connection [264]. The convergence of trigonometric series was also naturally considered about this time. We have seen that problems of TAYLOR and EULER in connection with a vibrating string called for trigonometric series. So also in other connections such series arose in works of BERNOULLIS, LEGENDRE, and LAPLACE. But FOURIER [246] was the first to assert and to attempt to prove (1812–1822) that any function, even

though for different values of the argument it is expressed by different analytical formulae, can be developed in such a series. Since the real importance of trigonometric series was thus shown by FOURIER for the first time, it was natural that his name should thereafter be associated with them [287].

In the nineteenth and early twentieth centuries considerable work was done in connection with two topics, of elementary geometry, namely: Geometrography, and, especially, the Geometry of the Triangle. Geometrography is a system for measuring "the coefficient of simplicity" and the "coefficient of exactitude" of constructions with ruler and compasses. It was initiated in 1888 and kept alive for a score of years by publications of ÉMILE LEMOINE, and many others [288].

We have already referred to EULER's Line, MALFATTI's Problem, Nine-Point Circle, WALLACE's Line—results in the Geometry of the Triangle, which was so extensively developed, especially after the impetus given by publications of LEMOINE and BROCARD 1873–1875. Thus a large new vocabulary was evolved [289].

Vector analysis as employed today originated mainly with two mathematical physicists, J. W. GIBBS [290], an American, 1881–1884, and OLIVER HEAVISIDE [291], an Englishman, 1891. Quaternions, an algebra of vector magnitudes, was developed earlier, about 1853, by that Irish genius, W. R. HAMILTON [292], who also contributed notably in the fields of dynamics and optics.

We have already noted that the subject of determinants originated with LEIBNIZ. LAPLACE, LAGRANGE, GAUSS, CAUCHY, JACOBI, and many others contributed to the development of the subject before 1841, as indicated in the first volume of the late Sir THOMAS MUIR's monumental work [219]. Four other volumes of a similar size were necessary for setting forth the history of the subject during the next eighty years.

For several centuries Mathematical Tables have notably contributed to advances in science, and in latest years the development of tabular material and of computing machines has been enormous. During the past decade an international Committee on Mathematical Tables and Other Aids to Computation, of the Division of Mathematical and Physical Sciences of the United States National Research Council, has been especially active, and for six years has published not only a quarterly periodical [293] which, throughout the world, disseminates information dealing with tabular material old and new, as well as with other computational aids, but also a Report on tables in the Theory of Numbers [294]. Moreover in this same decade the United States National Bureau of Standards has been directing a computing group [295] (at one time numbering 350, working in two shifts, 15 hours a day, for five days a week, with 169 machines) which has already published more than a score of valuable volumes of tables.

But from 1871–1948 the British Association for Advancement of Science has had a Mathematical Tables Committee, constantly arranging for the

calculation and publication of new tables, or reprints of old ones, mainly in the field of BESSEL functions. Thus, apart from tables published in annual *Reports* (1877–1929), 10 separate volumes (one in two editions) have been published (1900, 1931–1946). The Committee's first publication (1873) was a valuable annotated catalogue by **J. W. L. GLAISHER** of existing mathematical tables [**296**]. Its first published volume of tables, by CUNNINGHAM [**302a**], was followed by three volumes of factor tables by J. GLAISHER [**302**]. Since June 1948 the B.A. Committee has been under the direction of the Royal Society.

Under other auspices the British also published (i), Cambridge, 1926, a detailed report on various published tables of *Logarithms of Numbers* [**297**], and (ii), London, 1946, an exceedingly valuable *Index of Mathematical Tables* [**298**] listing not only those great ones incidentally already mentioned in our survey, such as: tables of RHETICUS and PITISCUS, BRIGGS, DASE, and LEGENDRE,—some of them not yet replaced—but also a host of others published and unpublished.

The BRIGGS' (1624) tables of logarithms of numbers 1(1)20000, 90000(1)101000 to 14 places of decimal, and the 10 decimal tables of VLACQ have been replaced (1924–1949) by the magnificent 20D tables of A. J. THOMPSON [**174**]. The RHETICUS (1596) 10D tables of natural trigonometric functions, for interval 10″, were elaborated (1911–1916) in ANDOYER's tables [**299**], 14–17D, and the superb factor tables and table of primes of D. N. LEHMER [**300**] (1909, 1914) render unnecessary the earlier works of BURCKHARDT [**301**] (1814–1817), J. GLAISHER [**302**] (1879–1883), DASE, (1862–1865), and others. The BRIGGS' 15D table of natural sines at interval 0°.01 (1633) was recalculated and published with central second differences (1949) by the National Bureau of Standards, but other tables of BRIGGS (1633) have not been replaced.

The mathematical tabular field is exceedingly extensive and is being constantly developed. Historical summaries in special fields serve a very useful purpose.

In conclusion we may indicate a few samples of references to historical accounts of developments of higher mathematics. E. W. BROWN [**303**] discussed the history of mathematics to the end of the second decade of the twentieth century. The lectures on the development of mathematics in the nineteenth century by KLEIN [**304**] are masterly. COURANT has discussed RIEMANN and mathematics of the last hundred years [**305**]. The development of applied mathematics in the nineteenth century was surveyed by R. S. WOODWARD [**306**]. The general sketches of BELL [**5**] and STRUIK [**6**] will also be found of interest. American activity 1888–1938 is indicated in the excellent *Semicentennial Addresses of the American Mathematical Society* [**307**] and in the biographies and bibliographies of all its presidents and secretaries published in its *History* [**308**]. Most of SMITH & GINSBURG's history [**309**] is devoted to the nineteenth century mathematics in America, and a certain amount of reference to higher mathematics may be found in CAJORI's *Teaching and History of Mathematics in*

the United States [310]. Then there is also the comparatively recent *A History of Science in Canada* [311]. Biographies of other nineteenth century mathematicians are to be found in works of BELL [53] and PRASAD [254] to which we have already made reference. Various histories of geometrical developments have been set forth by COOLIDGE [245], KÖTTER [275] and LORIA [312]. The 30 volumes of the American Mathematical Society's *Colloquium Lectures* and *Colloquium Publications* (1896–1948) are valued treatments of special modern topics of importance. Along with these may be noted the highly important 43 titles (1908–1930) of the *Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, and 54 volumes (1921–1947) of *Grundlehren der mathematischen Wissenschaften*, the briefer 25 *Ergebnisse der Mathematik und ihrer Grenzgebiete* (1932–1942), 105 *Mémorials des Sciences Mathématiques* (1925–1947), and the 39 *Cambridge Tracts in Mathematics and Mathematical Physics* (1905–1946).

We may conclude these lectures with recalling a passage more true today than when spoken by ISAAC BARROW in his oration of more than two and three-quarter centuries ago: *The Mathematics . . . is the fruitful Parent of . . . Arts, the unshaken Foundation of Sciences, and the Plentiful Fountain of Advantage to Human Affairs.*

LITERATURE LIST AND NOTES

Some of the more extensive general and original histories of mathematics in English are the following (LLN 1–5):

1. W. W. R. BALL, *A Short History of Mathematics*, fifth ed. London, 1912, xxiv, 522 pp. Although this work is much out of date and filled with erroneous statements, it is still the most readable of old-style histories in English.

2. F. CAJORI, *A History of Mathematics*, second ed., rev. and enl. New York, 1919. x, 514 pp. Includes the nineteenth and twentieth centuries.

3. F. CAJORI, *A History of Elementary Mathematics*, second ed., rev. and enl., New York, 1917 or 1921, viii, 324 pp. R. C. ARCHIBALD, "Florian Cajori (1859–1930)," *Isis*, vol. 17, 1932, pp. 384–407+portrait plate.

4. D. E. SMITH, *History of Mathematics*, corrected and rev. ed. Boston, v. 1, 1928, xxii, 596 pp.; v. 2, 1930, xxii, 725 pp.; there are changes in more than 100 places in v. 1 of the first ed. Many illustrations and portraits. The second vol. contains the topical history, and also many illustrations. W. D. REEVE, "David Eugene Smith," *Scripta Math.*, v. 11, 1945, pp. 209–212+2 portraits. Also anonymous biographical notes, pp. 364–369+4 portraits. Also "Bibliography of the critical, historical, and pedagogical writings of D. E. S.," *Osiris*, v. 1, 1916, pp. 9–78 and "The D. E. S. mathematical library of Columbia University" by BERTHA M. FRICK, pp. 79–84. Also L. G. SIMONS, "D. E. S.—In memoriam," Amer. Math. Soc., *Bull.*, v. 51, 1945, pp. 40–50.

5. E. T. BELL, *The Development of Mathematics*, second ed. enl. New York, 1945, xiv, 637 pp. A most readable and original modern exposition.

6. D. J. STRUIK, *A Concise History of Mathematics*. Volume I: *The Beginnings—The Beginnings in Western Europe*. Volume II: *The Seventeenth Century—The Nineteenth Century*. Paged continuously, New York, 1948, xxvii, 300 pp. More than 40 pages are devoted to portraits and other illustrations. Quotation: "We hope that . . . we have been able to give a fairly honest description of the main trends in the development of mathematics throughout the ages and of the social and cultural setting in which it took place."

7. O. E. NEUGEBAUER, "History of Mathematics," *Encyclopedia* (?), New York, P. Collier & Sons Corp., 1949?; 6000-word article.

7a. O. E. NEUGEBAUER, "Mathematical methods in ancient astronomy," Amer. Math. Soc., *Bull.*, v. 54, Nov. 1948, pp. 1013–1041.

7b. O. ORE, *Number Theory and its History*. New York, 1948, x, 370 pp. Attractively written v. in 15 chapters based on a course given for several years primarily for juniors and seniors at Yale University.

Among general histories of science, with most important mathematical sections, is the monumental and outstanding

8. G. SARTON, *Introduction to the History of Science. Volume I, from Homer to Omar Khayyam*, xii, 839 p.; *volume II, from Rabbi ben Ezra to Roger Bacon*, xxxviii, 1251 pp.; *volume III, Science and Learning in the Fourteenth Century*, xxxv, 1018, xiv, 1019–2155 pp. Washington, D. C., 1927, 1931, 1947–1948.

9. G. SARTON, *The Study of the History of Mathematics*. Cambridge, Mass., 1936. iv, 112 pp. Essay on the study of the history of mathematics, pp. 3–28; Note on the study of the history of modern mathematics, pp. 29–38; Bibliography, pp. 39–98.

10. J. W. N. SULLIVAN, *The History of Mathematics from the Fall of Greek Science to the Rise of the Conception of Mathematical Rigour*. London, 1925, 109 p. A very pleasantly written volume covering about 1000 years.

11. J. TROPFKE, *Geschichte der Elementar-Mathematik in systematischer Darstellung*. Berlin, 7 v., v. 1, third ed. 1930, viii, 222 pp.; v. 2, third ed., 1933, iv, 266 pp.; v. 3, third ed., 1937, iv, 239 pp.; v. 4, third ed., 1940, iv, 316 pp., edited after Tropfke's death by K. VOGEL; v. 5, second ed., 1923, iv, 185 pp.; v. 6, second ed., 1924, iv, 169 pp.; v. 7, second ed., 1924, iv, 128 pp.–Index v. A most admirable up-to-date history, rich in bibliographic references. J. E. HOFMANN, "Johannes Tropfke," *Deutsche Math.*, v. 6, 1941, pp. 114–118+1 plate (with list of publs.).

12. M. CANTOR, *Vorlesungen über Geschichte der Mathematik*. Leipzig, v. 1, third ed., vi, 942 pp.; v. 2, second ed., 1900, xii, 943 pp.; v. 3, second ed., 1901, x, 923 pp.; v. 4, 1908, vi, 1113 pp. in cooperation with others. The great history of mathematics (to the close of the eighteenth century), which, even with the important corrections in the historical periodical, *Bibliotheca Mathematica*, while offering many things of great value, leaves much to be desired in accuracy, and is now largely out of date. F. CAJORI, "Moritz Cantor, the historian of mathematics," Amer. Math. Soc., *Bull.*, v. 27, 1920, pp. 21–28. D. E. SMITH, "Moritz Cantor," *Scripta Math.*, v. 1, pp. 204–207, 1932.

13. Students of Babylonian mathematics will find the following popular work of special interest: (a) *They Wrote on Clay. The Babylonian Tablets Speak Today*. By E. CHIERA, ed. by G. G. CAMERON. Chicago, 1939, xv, 234 pp. Some other references are: (b) R. C. ARCHIBALD, (i) "Babylonian Mathematics with special reference to recent discoveries," *Mathematics Teacher*, v. 29, 1936, pp. 209–219; (ii) "Babylonian Mathematics," *Isis*, v. 26, 1936, pp. 63–81. (c) F. THUREAU-DANGIN, *Esquisse d'une Histoire du Système Sexagesimal*. Paris, 1932, 81 pp. (d) English ed. of (c) by S. GANDZ, rev. and enl. by the author, "Sketch of a history of the sexagesimal system," *Osiris*, v. 7, 1939, pp. 95–141. (e) V. G. CHILDE, "The oriental background of European science," *Modern Quarterly*, v. 1, 1938, pp. 105–120.

14. O. NEUGEBAUER, *Mathematische Keilschrift-Texte*, 3 v. *Quellen u. Studien zur Geschichte d. Math.*, v. 3A, 1935–1937, xii, 516 pp.; iv, 64 pp.+69 plates; viii, 84 pp.+3 plates. This, and the works listed in [**15–17**], are fundamental among those dealing with Babylonian mathematical tablets. For a biographical sketch and portrait of NEUGEBAUER, see *National Mathematics Mag.*, v. 11, pp. 14–16, 1936.

15. F. THUREAU-DANGIN, *Textes Mathématiques Babyloniens transcrits et traduits*. Leyden, 1938, xi, 243 pp.

16. O. NEUGEBAUER & A. SACHS, *Mathematical Cuneiform Texts* . . . with a Chapter by A. GOETZE (*Amer. Oriental Series*, v. 29). New Haven, Conn., 1945. x, 177 pp.+49 pages of plates. Only a very few of the 191 mathematical cuneiform tablets discussed in this volume are not in the United States. Among these tablets and those listed in the work of [**14**], 103 tablets at Yale University are elucidated. This is the most valuable lot of problem-texts in any existing collection. At the University of Pennsylvania are about 300 tablets (of which 100 have been published) but they, as well as 200 mathematical texts in the Istanbul Museum, are nearly all table-texts.

17. O. NEUGEBAUER, *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, v. 1, *Vorgriechische Mathematik*. Berlin, 1934, 224 p. Egyptian Mathematics is discussed only on pp. 110–165.

18. *Illustrated London News*, v. 173, 14 July 1928; and v. 175, 31 August 1929.

19. V. G. CHILDE, *What Happened in History*. (*Pelican Books*), New York, 1946, viii, 280 pp. An excellent history of civilization with numerous references to mathematics and astronomy. Bronze and Iron Ages are periods of early mathematics.

20. O. NEUGEBAUER, "Das Pyramidenstumpf-Volumen in der vorgriechischen Mathematik," *Quellen u. Studien zur Geschichte d. Math.*, v. 2B, pp. 347–351, 1933; K. VOGEL, "Eine Pyramidenstumpf-Aufgabe bei den Babyloniern," *Arch. Orientforschung*, v. 8, 1933, pp. 220–221. Compare [**37**].

21. K. VOGEL, "Zur Berechnung der quadratischen Gleichungen," *Unterrichtsblätter f. Math. u. Naturw.*, v. 39, 1933, pp. 76–81. S. GANDZ, "The origin and development of the quadratic equations in Babylonian, Greek, and early Arabic algebra," *Osiris*, v. 3, 1937, pp. 405–557. See also [**13**] and TROPFKE [**11**], v. 3, pp. 50–56.

22. T. L. HEATH, *A Manual of Greek Mathematics*, Oxford, 1931, xvi, 552 pp. "Ancient Babylonian mathematics," pp. 522–530. This work is an abridgment, brought up to date, of Heath's valuable, but somewhat out-dated, *History of Greek Mathematics*, 2 v., Oxford, 1921, xxviii, 1032 pp.

23. H. S. SCHUSTER, "Quadratische Gleichungen der Seleukidenzeit aus Uruk," *Quellen u. Studien* . . . , v. 1B, pp. 194–200. O. NEUGEBAUER, *Quellen u. Studien* . . . , v. 3B, pp. 247–248, 1936. K. VOGEL, "Bemerkungen zu den quadratischen Gleichungen der babylonischen Mathematik," *Osiris*, v. 1, 1936, pp. 703–717. TROPFKE [**11**], v. 3, pp. 54–56.

24. NEUGEBAUER & SACHS [**16**], p. 130.

25. O. NEUGEBAUER, *Quellen u. Studien . . .*, v. 1B, 1930, p. 124 and F. THUREAU-DANGIN, "Le prisme mathématique AO 8862," *Rev. d'Assyr.*, v. 29, 1932, pp. 1–10, 89–90. See also THUREAU-DANGIN [34], pp. 307–311.

26. See NEUGEBAUER [17], pp. 193–197; NEUGEBAUER [14], v. 3A, part 1, pp. 210–211, 215–216. O. NEUGEBAUER, "Über die Lösung kubischer Gleichungen in Babylonien," Gesell. d. Wissen., Göttingen, *Math.-phys. Kl., Nachr.*, 1933, pp. 316–321. K. VOGEL, "Kubische Gleichungen bei den Babyloniern?" Bayer. Akad. d. Wissen., *Math.-naturw. Kl., Sitzb.*, 1934, pp. 87–94.

27. *Quellen u. Studien . . .*, v. 2B, p. 304, 1932.

28. O. NEUGEBAUER, *Quellen u. Studien . . .*, v. 4B, 1937, pp. 82–85, also paragraph 61.

29. A. J. SACHS, "Reciprocals of regular sexagesimal numbers," *Jn. of Cuneiform Studies*, v. 1, 1947, pp. 219–240.

30. O. NEUGEBAUER, "The history of ancient astronomy; problems and methods," *Jn. of Near Eastern Studies*, v. 4, 1945, pp. 1–38; reprinted in enlarged form in Astron. Soc. of the Pacific, *Publs.*, v. 58, 1946, pp. 17–43, 104–142. This up-to-date monograph is of extraordinary interest and value for both the astronomer and the mathematician, not only on account of the clarity and breadth of its sweep, but also on account of the very extended list of references (170+) to the literature of the subject. Since this list is readily accessible, we shall not make specific reference to a number of items of the list as we might otherwise have done. A few supplementary titles are given in [31].

31. J. K. FOTHERINGHAM, "The indebtedness of Greek to Chaldean astronomy," *The Observatory*, v. 51, 1928, pp. 301–315; reprinted with comments by O. NEUGEBAUER, *Quellen u. Studien . . .*, v. 2B, 1932, pp. 28–44. B. L. VAN DER WAERDEN, "Egyptian 'Eternal Tables,' " Akad. v. Wetens., *Proceedings*, v. 50, 1947, pp. 536–547 and pp. 782–788, with special reference to NEUGEBAUER'S "Planetary Texts." Also, E. J. KNUDTZON & O. NEUGEBAUER, "Zwei astronomische Texte," Soc. R. d. Lettres de Lund, *Bull.*, 1946–1947, II, 1947, pp. 77–88+2 plates.

32. NEUGEBAUER [14], v. 3A, part 1, pp. 96–103, and part 2, plate 1; also H. WASCHOW & O. NEUGEBAUER, "Reihen in der babylonischen Mathematik," *Quellen u. Studien . . .*, v. 2B, pp. 298–303, 1932.

33. See NEUGEBAUER & SACHS [16], p. 43; this very exact value for $\sqrt{2}$ is also found in PTOLEMY'S Almagest (Heiberg ed., v. 1, chap. 10, p. 35, l. 16). See also [13] (b) (i). O. NEUGEBAUER, "Über die Approximation irrationaler Quadratwurzeln in der babylonischen Mathematik," *Arch. Orientforschung*, v. 7, 1931, pp. 90–99.

34. For a good example of such proof see SACHS [29], pp. 237–240. What underlies the formation of the Yale series texts and such solutions as are referred to in [23], are notably suggestive in this connection. Reference may also be given to an exceedingly interesting paper on algebra before 1000 A.D. F. THUREAU-DANGIN, "L'origine de l'algèbre," Acad. d. Inscriptions & Belles-Lettres, Paris, *Comptes Rendus, Bull.*, Jan.–June, 1940, pp. 292–318.

35. R. C. ARCHIBALD, "Mathematics before the Greeks," *Science*, n.s., v. 71, 1930, pp. 109–121, 342, and v. 72, 1930, p. 39. Both Babylonian and Egyptian mathematics are here surveyed. See also BELL [5], pp. 26–48; and O. NEUGEBAUER, "Über vorgriechische Mathematik," Hamburg Univ., Math. Sem., *Abh.*, v. 7, pp. 107–124.

36. J. E. QUIBELL, Hierakonopolis. London, 1900, plate 26B.

37. N. F. WHEELER, "Pyramids and their purpose," *Antiquity*, Gloucester, Engl., v. 9, 1935, pp. 5–21, 161–189 (pyramid at Gizeh), 292–304 (pyramid mysticism and mystification.) Alleged mathematical relations in the pyramids of Egypt are totally unfounded. HERODOTUS (−450), *History*, English translation A.D. GODLEY, (Loeb Classical Library), London, v. 2, 1921 pp. 424–427. WHEELER ridicules Herodotus along with the "pyramidiots." J. H. COLE, *Determination of the Exact Size and Orientation of the Great Pyramid of Giza*. Survey of Egypt, *Paper* no. 29, Cairo, 1925, 9 pp.+1 plan. L. BORCHARDT, *Längen und Richtungen der vier Grundkanten der grossen Pyramide bei Gise*. Berlin, 1920, ii, 21 pp.+5 plates. L. BORCHARDT, *Einiges zur dritten Bauperiode der grossen Pyramide bei Gise . . . mit einer Bemerkung zur zweiten Bauperiode der dritten Pyramide* by HERBERT RICKE. Berlin, 1932, 22 pp.+12 plates. An excellent recent work is LESLIE GRINSELL, *Egyptian Pyramids*. Gloucester, England, 1947, 195 pp.+1 plate.

38. R. ENGELBACH, *The Problem of the Obelisks from a Study of the Unfinished Obelisk at Aswan*, New York, 1923, 134 pp. S. CLARKE & R. ENGELBACH, *Ancient Egyptian Masonry*. London, 1930, xvi, 242 pp.

39. O. NEUGEBAUER, "Studies in ancient astronomy, VIII. The water clock in Babylonian astronomy," *Isis*, v. 37, 1947, pp. 37–43. R. W. SLOLEY, "Primitive methods of measuring time, with special reference to Egypt," *Jn. Egypt. Archeology*, v. 17, 1931, pp. 166–178+7 plates. A. POGO, "Egyptian water clocks," *Isis*, v. 25, 1936, pp. 403–425. L. BORCHARDT, *Die altägyptische Zeitmessung*, Berlin-Leipzig, 1920, x, 70 pp.+18 plates. F. THUREAU-DANGIN, "Clepsydre Babylonienne et Clepsydre Egyptienne," *Revue d'Assyriologie*, v. 30, 1933, pp. 51–52. See also ALBERT REHM, (a) "Horologium-Wasseruhren," PAULY'S *Real-Encyclopädie d. class. Altertumswissen.* new ed. G. WISSOWA, v. 8, Stuttgart, 1913, cols. 2428–2433; (b) "Neue Beiträge zur Kenntnis der antiken Wasseruhren," Bayer. Akad. d. Wissen., *Phil. hist. Kl., 1920*, no. 17, 1921, 27 pp. with 5 illustrs. Also THALHEIM, article "κλεψύδρα," Pauly's *Real Encycl.* v. 11_1, 1921, cols. 807–809. The word clepsydra is derived from two Greek words meaning to steal water. The clepsydra was introduced into Greece in the time of Plato and later into Italy; see M. C. P. SCHMIDT, *Kulturhistorische Beiträge zur Kenntnis des Griech. u. Röm. Altertums. II: Die Entstehung der antiken Wasseruhr*. Leipzig, 1912, vi, 113 pp., 13 plates. A water clock of bronze inlaid with gold was presented by the King of Persia to CHARLEMAGNE in 807. Clepsydrae are operated even today by the Chinese, in Canton, for example; see D. J. MAGOWAN, "Modes of keeping time known among the Chinese," (a) *Chinese Repository*, Canton, v. 20, 1851, pp. 426–432; reprinted in (b) *Smithsonian Report 1891*, pp. 607–612. Also S. COULING, *The Encyclopaedia Sinica*, Shanghai, 1917. See also *Fir-Flower Tablets, Poems from the Chinese* by FLORENCE AYSCOUGH, *English Versions* by AMY LOWELL, Boston, 1921, pp. 11, 180–181. Perhaps the best historical article in English on the clepsydra is in F. J. BRITTEN, *Old Clocks and Watches & their Makers*, 6th ed., London, 1932, pp. 9–14, 7 illustrs. See also **[314]**.

40. C. BOREUX, *Études de Nautique Égyptienne. L'Art de la Navigation en Égypte jusqu'à la Fin de l'Ancien Empire*. Cairo, 1925, vii, 569 pp.+plates.

41. A. NEUBURGER, *The Technical Arts and Sciences of the Ancients*, transl. by H. L. BROSE. New York, 1930, xxii, 518 pp. H. P. & M. W. VOWLES, *The Quest for Power from Prehistoric Times to the Present Day*. London, 1931. xv, 354 pp., especially pp. 1–100. J. H. BREASTED, "The earliest civilization and its transition to Europe," *Scientific Mo.*, v. 10, 1920, pp. 87–105, 183–209, 249–255; fully illustrated.

42. *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau*. Edited with Commentary by W. W. STRUVE, with the use of a hieroglyphic transcription by B. A. TURAEV. *Quellen u. Studien . . .*, Berlin, v. 1A, 1930, xii, 198 pp.+10 folding plates.

43. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Introduction, Transcription, Translation and Commentary* by T. E. PEET. London, 1923, iv, 136 pp.+23 plates.

44. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Volume 1: Free Translation and Commentary* by A. B. CHACE with the assistance of H. P. MANNING. *Bibliography of Egyptian Mathematics* by R. C. ARCHIBALD. *Volume II: Photographs, Transcription, Transliteration, Literal Translation* by A. B. CHACE, L. BULL & H. P. MANNING. *Bibliography of Egyptian and Babylonian Mathematics (Supplement)* by R. C. ARCHIBALD. *The Mathematical Leather Roll in the British Museum* by S. R. K. GLANVILLE. Oberlin, Ohio, 1927, x, 210 pp.; 1929, xvi, 126 plates, 20 pp. Almost every entry of the Bibliography has detailed annotations. With regard to the Leather Roll, see: S. R. K. GLANVILLE, "The math. leather roll in the British Museum," *Jn. Egypt. Arch.*, v. 13, 1927, pp. 232–238+4 plates. A. SCOTT & H. R. HALL, "Laboratory notes: Egyptian leather roll of the seventeenth century B.C.," *British Museum Quarterly*, v. 2, 1927, pp. 56–57+1 plate. O. NEUGEBAUER, "Zur ägyptischen Bruchrechnung," *Z. f. ägypt. Sprache*, v. 64, 1929, pp. 44–48; K. VOGEL, "Erweitert die Lederrolle unsere Kenntnis ägyptischer Mathematik?" *Archiv f. Gesch. d. Math. . . .*, v. 11, 1929, pp. 386–407.

45. B. GUNN & T. E. PEET, "Four geometrical problems from the Moscow mathematical papyrus," *Jn. Egypt. Arch.*, v. 15, 1929, pp. 167–185+1 plate. K. VOGEL, "The truncated pyramid in Egyptian mathematics," *Jn. Egypt. Arch.*, v. 16, 1930, pp. 242–249. T. E. PEET, "A problem in Egyptian geometry," *Jn. Egypt. Arch.*, v. 17, 1931, pp. 100–106+1 plate, and 154–160. O. NEUGE-

BAUER, "Die Geometrie der ägyptischen mathematischen Texte," *Quellen u. Studien . . .*, v. 1B, pp. 413–451. See also NEUGEBAUER [17], pp. 122–137. Compare [20].

46. BELL [5], "Firmly established Greece 600 B.C.–A.D. 300," pp. 49–84; SARTON [8], v. 1, pp. 65–413; HEATH [22].

46a. A work of reference of first importance and interest for the student is *Selections Illustrating the History of Greek Mathematics with an English Translation* by IVOR THOMAS, *in two volumes*, I: *From Thales to Euclid;* II: *From Aristarchus to Pappus*. (*Loeb Classical Library*), London, and Cambridge, Mass., 1939, xvi, 506 pp.; 1941, xii, 683 pp. Translations are given facing the Greek texts. There is an excellent subject index for the two volumes.

46b. The fundamental Bibliography of ancient mathematics during 1873–1930 has been published in 8 v. of *Jahresb. ü. d. Forts. d. kl. Altertumswissen.* The last vol. "Antike Mathematik," for the period 1906–1930, by CLEMENS TAER, filled v. 283, 1932, 144 pp. There are 498 entries, each with descriptive and critical notes, and the contents. The entries deal with mathematics before 600 A.D. The other 7 surveys are as follows: v. 1, 1873, "Jahresb. ü. d. exacten Wissen. bei den Griechen u. Römern," by B. LANGKAVEL, pp. 680–720; mathematics and astronomy, pp. 681–687. V. 11, 1879, "Jahresb. ü. Math. Astron. u. Mechanik im Alterthum für 1873–1877," by M. CURTZE, pp. 159–217. V. 23, 1882, by M. CURTZE, 1878–1879, pp. 217–242. V. 40, 1886, by M. CURTZE, 1879–1882, pp. 1–50 h. V. 108, 1902, by W. SCHMIDT, 1890–1901, pp. 59–128. V. 124, 1905, by J. L. HEIBERG, 1902–1905, pp. 129–143 (very brief). V. 129, 1907, by K. TITTEL, 1902–1905, pp. 113–219.

47. T. L. HEATH, "Mathematics and astronomy," in *The Legacy of Greece*, ed. by R. W. LIVINGSTONE, Oxford, 1922, pp. 97–136.

48. T. L. HEATH, "Greek mathematics and science," *Mathematical Gazette*, v. 10, 1921, pp. 289–301.

49. T. L. HEATH, "Greek geometry with special reference to infinitesimals," *Mathematical Gazette*, v. 11, pp. 249–259.

50. J. KLEIN, "Die griechische Logistik und die Entstehung der Algebra," *Quellen u. Studien . . .*, v. 3B, 1934, pp. 18–105; K. VOGEL, "Beiträge zur Griechischen Logistik," Bay. Akad. d. Wissen., *Math.-Naturw. Abt., Sitzungsb.*, 1936, pp. 357–472.

51. H. W. TURNBULL, *The Great Mathematicians*, London, 1929. Contents: THALES, PYTHAGORAS and the Pythagoreans, pp. 1–15; EUDOXUS and the Athenian School, pp. 16–29; Alexandria: EUCLID, ARCHIMEDES and APOLLONIUS, pp. 30–41: PAPPUS and DIOPHANTUS, pp. 42–53; NAPIER and KEPLER, pp. 54–69; DESCARTES and PASCAL, pp. 70–85; NEWTON, pp. 86–94; BERNOULLIS and EULER, pp. 95–100; MACLAURIN and LAGRANGE, pp. 101–107; GAUSS and HAMILTON, pp. 108–118.

52. W. W. R. BALL, "Pythagoras," *Mathematical Gazette*, v. 8, 1915, pp. 5–12. See also [51, 54] and Frank [57].

53. E. T. BELL, *Men of Mathematics*, New York, 1937, xxii, 593 pp. Extraordinarily readable and valuable volume. Contents: ZENO, EUDOXUS, ARCHIMEDES (pp. 19–34); DESCARTES (35–55); FERMAT (56–72); PASCAL (73–89); NEWTON (90–116); LEIBNIZ (117–130); the BERNOULLIS (131–138); EULER (139–152); LAGRANGE (153–171); LAPLACE (172–182); MONGE and FOURIER (183–205); PONCELET (206–217); GAUSS (218–269); CAUCHY (270–293); LOBATCHEVSKY (294–306); ABEL (307–326); JACOBI (327–339); HAMILTON (340–361); GALOIS (362–377); SYLVESTER and CAYLEY (378–405); WEIERSTRASS and SONJA KOWALEWSKI (406–432); BOOLE (433–447); HERMITE (448–465); KRONECKER (466–483); RIEMANN (484–509); KUMMER and DEDEKIND (510–525); POINCARÉ (526–554); CANTOR (555–579).

54. *Portraits of Eminent Mathematicians, with Brief Biographical Sketches*, by D. E. SMITH, I, II, New YORK, 1936–38; I: ARCHIMEDES, COPERNICUS, VIÈTE, GALILEO, NAPIER, DESCARTES, NEWTON, LEIBNIZ, LAGRANGE, GAUSS, LOBACHEVSKY, SYLVESTER; II: EUCLID, CARDAN, KEPLER, FERMAT, PASCAL, EULER, LAPLACE, CAUCHY, JACOBI, HAMILTON, CAYLEY, CHEBYSHEV, POINCARÉ. *Portraits of Famous Philosophers who were also Mathematicians, with Biographical Accounts* by C. J. KEYSER, New York, 1939: PYTHAGORAS, PLATO, ARISTOTLE, EPICURUS, ROGER BACON, DESCARTES, PASCAL, SPINOZA, LEIBNIZ, BERKELEY, KANT, PEIRCE. Of course there are no known portraits of ARCHIMEDES, ARISTOTLE, EUCLID, PLATO. These three portfolios were published by *Scripta Mathematica*.

55. O. NEUGEBAUER & W. W. STRUVE, "Über die Geometrie des Kreises in Babylonien," *Quellen u. Studien . . .*, v. 1B, 1929, pp. 81–92.

56. According to IAMBLICHUS (c. 325). In E. B. ESCOTT, "Amicable numbers," *Scripta Math.*, v. 12, 1946, pp. 61–72, 390 pairs of amicable numbers are listed. See also *Math. Tables and Other Aids to Computation*, v. 1, pp. 95–96; v. 2, p. 211. In *Scripta Mathematica*, v. 14, Mar. 1948, p. 77 are "43 new couples of amicable numbers" by the late PAUL POULET.

57. R. C. ARCHIBALD, "Mathematicians and music," *Amer. Math. Mo.*, v. 31, 1924, pp. 1–25. See also the admirable work, E. FRANK, *Plato und die sogenannten Pythagoreer, ein Kapitel aus der Geschichte des griechischen Geistes*. Halle, 1923, x, 400 pp.

58. T. L. HEATH, (a) *Aristarchus of Samos, the Ancient Copernicus. A History of Greek Astronomy to Aristarchus together with Aristarchus's Treatise on the Sizes and Distances of the Sun and Moon.* Oxford, 1913, viii, 425 pp. In this work Heath's account of Greek astronomy is mainly derived from researches of GOMPERZ. More recent investigation [27] renders unreliable these researches, as well as references to Pythagorean music. In this connection see FRANK [57], with many references to literature. (b) *Greek Astronomy.* London, 1932, viii, 192 pp. Translations of astronomical passages in many Greek authors with an Introduction; (c) *The Copernicus of Antiquity (Aristarchus of Samos) (Pioneers of Progress).* London, 1920, iv, 59 pp.

59. There is a valuable discussion of this in W. A. HEIDEL, *The Frame of the Ancient Greek Maps with a Discussion of the Discovery of the Sphericity of the Earth.* New York, 1937, x, 141 pp. The first unequivocal evidence of the earth as a sphere is found in PLATO'S *Phaedo;* see also ARISTOTLE, *De Caelo*, I, §§5–12 and II, §§1, 4, 5, 6.

60. I. B. HART, *Makers of Science—Mathematics, Physics, Astronomy.* London, 1928, 320 pp., illustr.; ARISTOTLE pp. 19–32. KEYSER [54]. The first historian of mathematics, EUDEMUS, was a pupil of ARISTOTLE. He wrote, among other works, a History of Arithmetic, a History of Astronomy, and a History of Geometry.

61. HEATH [22], *History*, v. 1, pp. 284–315; BELL [5], p. 73; HEIDEL [59]; FRANK [57], the most valuable survey; H. CHERNISS, *The Riddle of the Early Academy.* Berkeley, Cal., 1945, viii, 103 pp.; SARTON [8], v. 1, p. 113 ff; W. A. HEIDEL, "The Pythagoreans and Greek mathematics," *Amer. Jn. Philology*, v. 61, 1940, pp. 1–33 (the writer seemed ignorant of FRANK [57]); J. BURNET, *Platonism*, Berkeley, 1928, "Plato's mathematics," pp. 96–112; F. SOLMSEN, "Plato's Einfluss auf die Bildung der mathematischen Methode," *Quellen u. Studien . . .*, v. 1B, pp. 93–107, 1929; S. DEMEL, "Platons Verhältnis zur Mathematik," *Forschung zur Geschichte der Philosophie und der Pädogogik*, Leipzig, v. 4, 1929, part 1, vi, 146 pp.; KEYSER [54].

62. Should we accept the suggestion of PROCLUS that the PYTHAGOREANS were the first to declare the four Elements (earth, fire, air, water) to be the material principles from which the universe was evolved? Certainly PLATO in his *Timaeus* was the first to associate these Elements with the Regular Solids: the earth arising from the cube, fire from the tetrahedron, air from the octahedron, and water from the icosahedron. The theory of the Elements is not related to the PYTHAGOREAN school but to DEMOCRITUS (–400). Such are important results in the following valuable book, clarifying the role of the PYTHAGOREANS in connection with mathematics, a role grossly exaggerated in antiquity: EVA SACHS, *Die Fünf Platonischen Körper zur Geschichte der Mathematik und der Elementenlehre Platons und der Pythagoreer (Philologische Untersuchungen*, v. 24). Berlin, 1917, x, 242 pp.

63. BELL [53]; H. D. P. LEE, *Zeno of Elea. A Text with Translations and Notes.* Cambridge Univ. Press, 1936, vi, 125 pp.; F. CAJORI, "The history of Zeno's arguments on motion: phases in the development of the theory of limits," *Amer. Math. Mo.*, v. 22, 1915, pp. 1–6, 39–47, 77–82, 109–115, 143–149, 179–186, 215–220, 253–258, 292–297; B. L. VAN DER WAERDEN, "Zenon und die Grundlagenkrise der griechischen Mathematik," *Math. Annalen*, v. 117, 1940, pp. 141–161.

64. BELL [5]; HEATH [49]; TURNBULL [51]; O. BECKER, "Eudoxus-Studien," *Quellen u. Studien . . .*, "I. Eine voreudoxische Proportionslehre und ihre Spuren bei Aristoteles und Euklid," "II. Warum haben die Griechen die Existenz der vierten Proportionale angenommen?," v. 2B, 1933, pp. 311–333, 369–387; "III. Spuren eines Stetigkeitsaxioms in der Art des Dedekind'schen zur Zeit des Eudoxus," v. 3B, 1936, pp. 236–244; "IV. Das Prinzip des ausgeschlossenen Dritten in der griechischen Mathematik"; "V. Die eudoxische Lehre von den Ideen und den Farben," v. 3B,

1936, pp. 370–410. HANS KÜNSSBERG, *Der Astronom, Mathematiker und Geograph Eudoxus von Knidos. I. Teil: Lebensbeschreibung des Eudoxus, Überblick über seine astronomische Lehre und geometrische Betrachtung der Hippopede* (Progr. Realschule, Dinkelsbühl, 1888, 59 pp.+folding plate). *II. Teil: Mathematik* (Progr. Realschule, Dinkelsbühl, 1890, 61 pp.+folding plate.) Eudoxus also did important work in geography and in explanations of planetary motion; see NEUGEBAUER [30], HEIDEL [59], pp. 95–102; and G. SCHIAPARELLI, *Scritti sulla Storia della Astronomia Antica*, Bologna, v. 2, 1926, pp. 2–112. In the outstanding work on the planetary theory by EUDOXUS he made an attempt to explain the peculiarities of a planetary movement known as retrogradation by the assumption of the superposition of the rotation of two concentric spheres around inclined axes and in opposite directions. In this way he reached a satisfactory explanation of the general type of planetary movement and thereby inaugurated in the history of astronomy a new period which was marked by attempts to explain the movements of the planetary system by mechanical models.

65. W. W. R. BALL, "Three classical geometrical problems," in his *Mathematical Recreations & Essays*, eleventh ed., rev. by H. S. M. COXETER, London, 1939, pp. 326–349. Also THOMAS [46a], see Index, under circle-squaring, cube-duplication, trisection of an angle. See also J. S. MACKAY, "The ancient methods for the duplication of the cube," Edinburgh Math. Soc., *Proc.*, v. 4, 1886, pp. 2–20; and H. DÖRRIE, *Triumph der Mathematik. Hundert berühmte Probleme aus zwei Jahrtausenden mathematischer Kultur*, Breslau, second ed., 1940, viii, 386 pp.

66. F. GOMES TEIXEIRA, *Sur les Problèmes Célèbres de la Géométrie Élémentaire non résolubles avec la Règle et le Compas*, Coimbra, 1915, also in his *Obras sobre Mathematica*, Coimbra, v. 7, 1915, pp. 285–412; F. ENRIQUES, ed., *Fragen der Elementargeometrie*, 2 Teil, second ed., Leipzig, 1923: Aufgaben dritten Grades: Verdoppelung des Würfels, Dreiteilung des Winkels," by A. CONTI, pp. 189–226; "Über die transzendenten Aufgaben, insbesondere über die Quadraturen des Kreises" by B. CALÒ, pp. 267–326. ORE [7b], "The classical construction problems," pp. 340–348.

67. E. W. HOBSON, *"Squaring the Circle," a History of the Problem*, Cambridge, 1913, viii, 58 pp.

68. F. KLEIN, *Famous Problems of Elementary Geometry, The Duplication of the Cube, the Trisection of an Angle, the Quadrature of the Circle*, transl. by W. W. BEMAN and D. E. SMITH. Second ed. rev. and enl. by R. C. ARCHIBALD, New York, 1930, xii, 92 pp.

69. R. C. ARCHIBALD, "Curves, Special," *Encyclopædia Britannica*, New York, fourteenth ed., v. 6, 1929, pp. 887–899.

70. All teachers of the history of Greek mathematics will want to refer their students to the excellent address of R. E. LANGER, "Alexandria—shrine of mathematics," *Amer. Math. Monthly*, v. 48, 1941, pp. 109–125.

71. TURNBULL [51]; SMITH [54]; HEATH [73], v. 1, pp. 1–240; A. DEMORGAN, article "Euclid" in W. SMITH, *Dictionary of Greek and Roman Biography*, v. 2, London, 1890, pp. 63–74; F. HULTSCH, "Eukleides," PAULY-WISSOWA, *Real-Encyclopädie*, v. 11, 1907, cols. 1003–1052.

72. EUCLID's ten treatises are 1. Elements, 2. Data, 3. Optics, 4. Phaenomena, 5. Music, 6. Pseudaria, 7. Surface-Loci, 8. Conics, 9. Porisms, 10. On Divisions of Figures. See HEATH [67], v. 1, pp. 7–18. Of 1–5 approximately complete texts have come down to us. In the case of 6–7 our fragmentary knowledge derived wholly from Greek sources makes conjecture as to their content rather futile. As to 10, PROCLUS alone among Greeks makes explanatory reference. But an Arabic ms. (translated into French) gives the enunciation of all of the propositions (36) and also the proofs of four of them. This and other documents were the basis of R. C. ARCHIBALD, *Euclid's Book on Division of Figures . . . with a Restoration based on Woepcke's Text and on the Practica Geometriae of Leonardo Pisano*. Cambridge, 1915, viii, 88 pp.

73. T. L. HEATH, *The Thirteen Books of Euclid's Elements translated from the text of Heiberg with Introduction and Commentary*, second ed. revised with additions. Cambridge, 3 vols., 1926, xii, 432 pp.+folding plate; vi, 436 pp., iv, 546 pp. This monumental edition of the *Elements* is also an encyclopedic history of elementary geometry. Who first laid down the criterion that the ruler and compasses shall be the only instruments used in elementary geometry, "plane" problems? In EUCLID's Elements these instruments are postulated as follows: 1. To draw a straight line from any point to any point; 2. To produce a finite straight line continuously in a straight line; 3. To describe a circle with any center and distance. See the doctoral dissertation, A. D. STEELE, "Über die Rolle

von Zirkel und Lineal in der griechischen Mathematik," *Quellen u. Studien . . .*, v. 3B, pp. 287–369, 1936. ORE [7b], "Euclid's algorism," pp. 41–49.

74. The first English edition of EUCLID's *Elements* was brought out at London in 1570, under the name of Sir HENRY BILLINGSLEY, later Sheriff and Lord Mayor of London. (For the small part which BILLINGSLEY may have had to do with the translation see A. DEMORGAN, *British Almanac and Companion for 1837*, pp. 38–39 of *Companion*). It contained "a very fruitful Praeface" by JOHN DEE, a man of great erudition; see W. F. SHENTON, "The first English Euclid," *Amer. Math. Monthly*, v. 35, 1928, pp. 505–512.

75. The *Phaenomena* consists of propositions in spherical geometry so far as required for observational astronomy. The earliest mathematical work which has come down to us entire is *On the Moving Sphere* by AUTOLYCUS of Pitane, an elder contemporary of EUCLID. Both works consider special circles on the heavenly sphere, the equator and parallel circles, the zodiac or ecliptic, and the horizon which here appears for the first time as a single word. But EUCLID's work seems to be based also on an unknown earlier work of more mathematical content. Of this work of AUTOLYCUS, and of another of his works, *On Risings and Settings*, as well as of the *Sphaerica* of the mathematician and astronomer, THEODOSIUS (bet. –180 and –25) of Bithynia, there is a handy German translation: *Autolykos Rotierende Kugel und Aufgang und Untergang der Gestirne. Theodosios von Tripolis* [sic] *Sphaerik, übersetzt und mit Anmerkungen versehen* (*Ostwald's Klassiker*, no. 232), Leipzig, 1931, viii, 180 pp. The work of THEODOSIUS, of the same general type as the other works mentioned above, is that of a laborious compiler offering practically nothing original. Such were the predecessors of the great work of MENELAUS.

76. The first English translation of EUCLID's *Optics*, by E. H. BURTON, was published in Amer. Optical Soc., *Jn.*, v. 35, 1945, pp. 357–372. EUCLID here proves the equivalent of the fact that $\tan \alpha/\tan \beta < \alpha/\beta$; and in another proposition it is assumed that the angles of incidence on a mirror and reflection therefrom are equal.

77. See ARCHIBALD [57], pp. 8–10, with various references to the literature.

78. HEATH [72], v. 1, pp. 202–220. L. R. LIEBER, *Non-Euclidean Geometry; or, Three Moons in Matheses*. Brooklyn, N. Y., 1931, 46 pp. D. E. SMITH, general ed., *A Source Book in Mathematics*. New York, 1929, xviii, 701 pp. +7 portrait plates. The 96 extracts are edited by many collaborators. R. BONOLA, *Non-Euclidean Geometry. A Critical and Historical Study of Its Development*. English translation by H. S. CARSLAW, Chicago, 1912, xii, 268 pp.

79. R. C. ARCHIBALD, (a) "Perfect numbers," *Amer. Math. Monthly*, v. 28, 1921, pp. 140–141; (b) "Mersenne's numbers," *Scripta Mathematica*, v. 3, pp. 112–119, 1935. There are 55 MERSENNE numbers which are defined by $M_n = 2^n - 1$, n prime, and less than 263. The largest known prime number (39 integers) is M_{127}, discovered in 1914. Regarding the M_n the following facts are now known: 12 (prime), 14 (composite and completely factored), 9 (two or more prime factors), 8 (only one prime factor), 12 (composite but no factor known). See *Math. Tables and Other aids to Computation*, v. 1, pp. 333, 404; v. 2, pp. 94, 341; and H. S. UHLER, Nat. Acad. Sci., *Proc.*, v. 34, 1948, pp. 102–103; and Amer. Math. Soc., *Bull.*, v. 54, 1948, pp. 378–380.

80. HARRIET H. SHOEN, "Archimedes. The reconstruction of a personality," *Scripta Math.*, v. 2, 1934, pp. 261–264, 342–347. T. L. HEATH, (a) *Archimedes* (*Pioneers of Progress*). London, 1920. (b) *The Works of Archimedes, Edited in Modern Notations with Introductory Chapters*. Cambridge, 1897. clxxxvi, 326 pp. The long and very valuable introduction to (b) gives a life of ARCHIMEDES, an account of (1) the relation of A. to his predecessors; (2) arithmetic and A.; (3) the problems known as "neuseis" (Vergings or Inclinations); (4) cubic equations; (5) anticipations by A. of the integral calculus. Then follow works of A.: (i) On the sphere and cylinder; (ii) Measurement of a circle; (iii) On conoids and spheroids; (iv) On spirals; (v) On the equilibrium of planes; (vi) The sand-reckoner; (vii) Quadrature of the parabola; (viii) On floating bodies; (ix) Book of lemmas; (x) The cattle problem. For a historical summary with regard to the cattle problem see R. C. ARCHIBALD, *Amer. Math. Monthly*, v. 25, 1918, pp. 411–414. In connection with (2) references may be given to C. MÜLLER, "Wie fand Archimedes die von ihm gegeben Näherungswerte von $\sqrt{3}$," *Quellen u. Studien . . .*, v. 2B, pp. 281–285, 1932, to [97]; and to G. A. GIBSON, "The treatment of arithmetic progressions by Archimedes," Edinburgh Math. Soc., *Proc.*, v. 16, 1897, pp. 2–12. Under the general heading of this note we refer also to BELL [53]; SMITH [54]; and to an extra-

ordinary ancient colored mosaic representing the manner of death of A. This mosaic came originally from the city of Herculaneum, destroyed by an eruption of Vesuvius in 79 A.D. A reproduction of the mosaic in colors is given in a publication of a Bonn archaeologist, F. WINTER, *Der Tod des Archimedes*. Berlin, 1924, 24 pp.+2 plates; see also F. CAJORI, *Science*, v. 61, 1925, pp. 415.

81. As a supplement to the edition of his *Works* of A., Heath published a translation from the Greek of *The Method of Archimedes Recently Discovered by Heiberg* . . . , Cambridge, 1912, 51 pp. An earlier English translation, from a German translation, was published in *The Monist*, April 1909, and reprinted as a pamphlet with an Introduction by D. E. SMITH, *Geometrical Solutions Derived from Mechanics*, Chicago, 109, ii, 28 pp. Discussion in the *Method* is also related to the topic treated in J. M. CHILD, "Archimedes' principle of the balance and some criticisms upon it," in C. SINGER, ed., *Studies in History and Method of Science*. Oxford, v. 2, 1922, pp. 490–520. A sort of "neusis" is used by Archimedes in his remarkable solution of the construction of a regular heptagon, found comparatively recently in an Arabic ms.; see *Die trigonometrischen Lehren des Persischen Astronomen Abu'l-Raihân Muh ibn Aḥmad* AL-BIRÛNÎ, *dargestellt nach al-Qânûn al-Mas'ûdi von* CARL SCHOY. *Nach dem Tode des Verfassers herausgegeben von* JULIUS RUSKA *und* HEINRICH WIELEITNER. Hannover, 1927, xii, 108 pp. Also C. SHOY, "Graeco-Arabische Studien nach Handschriften der Vizeköniglichen Bibliothek zu Kairo," *Isis*, v. 8, 1926, pp. 21–40; J. TROPFKE, "Die Siebeneckabhandlung des Archimedes," *Osiris*, v. 1, 1936, pp. 636–651.

82. HEATH [49]; H. WIELEITNER, "Das Fortleben der Archimedischen Infinitesimalmethoden bis zum Beginn des 17. Jahrh., insbesondere über Schwerpunktbestimmungen," *Quellen u. Studien* . . . , v. 1B, pp. 201–220, 1930; W. STEIN, "Der Begriff des Schwerpunktes bei Archimedes," *Quellen u. Studien* . . . , v. 1B, pp. 221–224, 1930. When the idea of center of gravity was first conceived by the Greeks is unknown; in writings of Archimedes it appears to be already assumed as familiar.

83. HEATH [22], *History*, v. 2, pp. 98–101; M. BRÜCKNER, *Vieleck und Vielflache Theorie und Geschichte*. Leipzig, 1900, pp. 132–140+plate VI (with pictures of all of the solids). See [107].

84. J. SCOTT, "On the burning mirrors of Archimedes, with some propositions relating to the concentration of light produced by reflectors of different forms," R. Soc. Edinburgh, *Trans.*, v. 25, 1868, pp. 123–149+plate III.

85. R. A. FISHER, "Reconstruction of the sieve of Eratosthenes," *Math. Gazette*, v. 14, pp. 565–566, 1929. In E. HOPPE'S *Mathematik und Astronomie im klassischen Alterthum* (Heidelberg, 1911), it is stated on p. 284 that the method of the sieve of ERATOSTHENES for finding prime numbers was set forth completely in section 52 of Plato's *Phaedo*. This statement has been quoted more than once. On consulting the section in question, I find not the slightest basis for the statement's substantiation.

86. The best first introduction to APOLLONIUS would be through *Apollonius of Perga, Treatise on Conic Sections, edited in Modern Notation with Introductions including an Essay on the Earlier History of the Subject* by T. L. HEATH. Cambridge, 1896, cxxii, 254 pp. The earlier history of conics includes the names MENAECHMUS, EUCLID, and ARCHIMEDES. APOLLONIUS, and his own account of the conics is discussed, pp. lxvii–lxxxvi. Then follow further HEATH surveys: (a) "The construction of a conic by means of tangents," pp. cxxx–cxxxvii; (b) "The three-line and four-line locus," pp. cxxxviii–cl; (c) "The construction of a conic through five points," pp. cli–clvi. In the *Conics* of APOLLONIUS are the essentials for the construction of all three conics by the method (a). It seems certain that APOLLONIUS was the first to solve the problem (b), although there are no propositions leading directly to the results. Assuming that APOLLONIUS was in possession of a complete solution of the problem of constructing the four-line locus referred to the sides of a quadrilateral of any form, he had in fact solved the problem of constructing a conic through five points. But the problem of the construction of a conic through five points is, however, not found in the work of APOLLONIUS any more than the actual determination of the four-line locus. The PAPPUS generalization of the four-line locus may thus be stated: If $p_1, p_2, p_3, \cdots, p_n$ be the lengths of straight lines to meet n given straight lines at given angles (where n is odd) and a be a given line, then if

$$\frac{p_1}{p_2}\cdot\frac{p_3}{p_4}\cdots\frac{p_n}{a}=\lambda,$$

where λ is a constant, the point will be on a curve given in position. This will also be true if n is even, and

$$\frac{p_1}{p_2}\cdot\frac{p_3}{p_4}\cdots\frac{p_{n-1}}{p_n}=\lambda.$$

See THOMAS [46a], v. 2, p. 603.

See also HEATH [22], *History*, v. 2, pp. 126–196. There is also the German edition of another excellent work, H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum*. Copenhagen, 1886, xiv, 511 pp. A monumental edition of the complete Conics of Apollonius, the eighth book restored, was published in 1710 by EDMOND HALLEY, to whom we shall later [211] return. Reference may also be given to O. NEUGEBAUER, "Apollonius—Studien (Studien zur Geschichte der antiken Algebra II.)," *Quellen u. Studien* . . . , v. 2B, pp. 215–253, 1932. See also O. NEUGEBAUER, "The astronomical origin of the theory of conic sections," Amer. Phil. Soc., *Proc.*, v. 92, 1948, pp. 136–138.

86a. See THOMAS [46a], v. 1, pp. 362–363, 495–496.

87. For example, C. HELLWIG, *Das Problem des Apollonius* . . . Halle, 1856, iv, 43 pp.+4 plates. B. ALVORD, *The Tangencies of Circles and Spheres* (*Smithsonian Contribs. to Knowledge*). Washington, 1856, 16 pp.+9 plates.

88. See R. C. ARCHIBALD, (a) "Centers of similitude and certain theorems attributed to Monge. Were they known to the Greeks?," *Amer. Math. Monthly*, v. 22, 1915, pp. 6–12; (b) "Historical note on centers of similitude of circles," *idem*, v. 23, 1916, pp. 159–161.

89. R. C. ARCHIBALD, "Discussion and history of certain geometrical problems of Heraclitus and Apollonius," Edinburgh Math. Soc., *Proc.*, v. 28, 1910, pp. 152–178+5 folding plates. A hundred solutions of the problems of Heraclitus are given in a notable book: A. MAROGER, *Le Problème de Pappus et ses cent Premières Solutions*. Paris, 1925, viii, 386 pp. For an account of the important work of APOLLONIUS in connection with epicycles and eccentrics see J. L. E. DREYER, *History of the Planetary Systems from Thales to Kepler*. Cambridge, England, 1906, pp. 151 f.

90. FOTHERINGHAM [31]; the exact dating here of these astronomers is without foundation, since the reasoning of SCHNABEL, upon which the statements are based, is entirely wrong.

91. HEATH [22], *History*, v. 2, pp. 253–260; *Manual*, pp. 393–399.

92. There is not the slightest evidence that the Babylonians observed the precession of the equinoxes. Various writers, such as F. CAJORI, "Babylonian discovery of the precession of the equinoxes," *Science*, n.s. v. 65, 1927, p. 184, were misled by the wholly erroneous monograph of P. SCHNABEL, "Kidenas, Hipparch und die Entdeckung der Präzession," *Z. v. Assyriologie*, n.s., v. 3, 1926, pp. 1–60.

93. H. VOGT, "Versuch einer Wiederherstellung von Hipparchs Fixsternverzeichnis," *Astron. Nach.*, v. 225, 1925, cols. 17–54.

94. O. NEUGEBAUER, *Quellen u. Studien* . . . , v. 2B, pp. 41–44, where both Babylonian and Greek sources are quoted. See also THOMAS [46a], v. 2, pp. 395–397; and F. THUREAU-DANGIN, "La division du cercle," *Revue d'Assyriologie*, v. 25, 1928, pp. 177–188 [see ARCHIBALD [44] (Supplement), 1928].

95. HEATH [22], *History*, v. 2, pp. 298–354; *Manual*, pp. 415–433.

96. A. ROME, "Le problème de la distance entre deux villes dans la Dioptra de Héron," Soc. Sci. de Bruxelles, *Annales*, v. 42, 1923, mémoires pp. 234–258; O. NEUGEBAUER, "Über eine Methode zur Distanzbestimmung Alexandria-Rom bei Heron," Danske Vidensk. Selskab., *hist.-filol. Meddelelser*, v. 26, no. 2, 1938, 26 pp.+5 plates. Part 2, v. 26, no. 7, 1939, 11 pp. NEUGEBAUER discusses the dating of HERON and makes it seem plausible that he flourished in the first century of the Christian era.

97. J. E. HOFMANN, "Über die Annäherung von Quadratwurzeln bei Archimedes und Heron," Deutsche Math.-Ver., *Jahresb.*, v. 43, pp. 187–210, 1934.

98. Attention may be drawn to an interesting summary article by J. TROPFKE, "Zur Geschichte der quadratischen Gleichungen über dreieinhalb Jahrtausend," Deutsche Mathem.-Ver., *Jahresb.*, v. 43, pp. 98–107, Dec. 1933; v. 44, pp. 26–47, June, and pp. 95–119, Aug. 1934. The sections

dealt with in these parts are as follows: I. Introduction. Survey of Known Results; II. Babylonian Mathematics; III. EUCLID; IV. HERON; V. DIOPHANTUS; VI. Muhammed ibn Mūsā AL-KHOWĀRAZMĪ; VII. ABŪ KĀMIL SHOJĀ; VIII. AL-KARKHĪ; IX. 'OMAR KHAYYAM; X. ABRAHAM BAR CHIJA, named SAVASORDA; XI. LEONARDO VON PISA; XII. Anonymous arrangement of the Algebra of AL-KHOWĀRAZMĪ; XIII. MICHAEL STIFEL; XIV. GIROLAMO CARDANO; XV. RAFAEL BOMBELLI; XVI. CHRISTOPH CLAVIUS; XVII. SIMON STEVIN.

99. Perhaps the best first introduction to DIOPHANTUS is T. L. HEATH, *Diophantus of Alexandria. A Study in the History of Greek Algebra. Second ed., with a Supplement containing an Account of Fermat's Theorems and Problems connected with Diophantine Analysis and some Solutions of Diophantine Problems by Euler.* Cambridge, 1910, viii, 387 p. See also HEATH [22], *History*, v. 2, pp. 448–517, and *Manual*, pp. 472–509.

100. J. KLEIN, *Quellen u. Studien . . .*, v. 3B, pp. 133–135, 1936. This is a long footnote about the date of DIOPHANTUS. See also Heath [22], *History*, v. 2, pp. 306; dedication to Dionysius.

101. The history of the development of this subject is most elaborately set forth in L. E. DICKSON, *History of the Theory of Numbers*, v. 2, *Diophantine Analysis*. Washington, D. C., 1920, xxxv, 803 pp. ORE [7b], "Diophantine problems," pp. 165–208.

102. HEATH [22], *History*, v. 2, pp. 260–273; *Manual*, pp. 399–402. The question as to whether MENELAUS prepared a Catalogue of fixed stars is discussed by A. A. BJÖRNBO, *Bibl. Math.*, v. 2, 1901, p. 196–212. See also DREYER [106], 1917, pp. 533–535.

103. A Latin edition of the *Sphaerica* based on Hebrew and Arabic mss. was prepared by EDMOND HALLEY and published posthumously in 1758; note HALLEY'S earlier Greek work in [86]. There are also the following two German translations based on a considerable body of Arabic material: (1) A. A. BJÖRNBO, *Studien über Menelaos' Sphärik, Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen. (Abh. z. Gesch. d. Math. Wissen.*, v. 14). Leipzig, 1902, viii, 154 pp. (2) M. KRAUSE, *Die Sphärik von Menelaus aus Alexandria. . . .* (Gesell. d. Wissen. z. Göttingen, *Abh., phil.-hist. Kl.*, s. 3, no. 17). Berlin, 1936, viii, 254, 112 pp.+7 plates. This latter edition is by far the best.

104. E. H. BUNBURY & C. R. BEVAN, *Encycl. Britannica*, eleventh ed., v. 22, 1911, pp. 618–626; HEATH [22], *History*, v. 2, pp. 273–297; *Manual*, pp. 402–414.

105. A. G. BURGESS, "Ptolemy's theorem and certain trigonometrical formulae," Edinburgh Math. So., *Math. Notes*, no. 4, 1910, pp. 42–43; *School Science and Math.*, v. 17, 1917, pp. 784–786; R. C. ARCHIBALD, "Ptolemy's theorem and formulae of trigonometry," *Amer. Math. Monthly*, v. 25, 1918, pp. 94–95. That PTOLEMY had given a solution of the so-called SNELL-POTHENOT Problem in connection with astronomy rather than geodesy, navigation, or elementary geometry, was recorded in J. A. C. OUDEMANS, (a) "Lösung des sog. Pothenot'schen, besser Snellius'schen Problems von Ptolemaeus," Astron. Gesell., *Vierteljahrschrift*, v. 22, 1887, pp. 345–349; (b) "Het problema van Snellius obgelost door Ptolemaeus," Akad. v. Wetens., *Verslagen, afd. Natuur.*, s. 2, v. 19, 1884, pp. 436–441. This problem has a very extended literature including discussions by C. F. GAUSS (1816–1852). The solution of WILLEBRORD SNELL (1580 or 1581–1626) was given in his *Eratosthenes Batavus de terrae ambitus vera quantitate*. Leyden, 1717, pp. 199–200. To him is also due (1624) the name "loxodrome" for the rhumb line in navigation, a rediscovery of the Hindu formula for the area of a circumscribed quadrilateral in terms of its sides, and the law of refraction of light (see J. A. VOLLGRAFF, "Snellius' notes on the reflection and refraction of rays," *Osiris*, v. 1, 1936, pp. 718–725.)

106. The best edition of this Catalogue is *Ptolemy's Catalogue of Stars, a Revision of the Almagest*, by C. H. F. PETERS and E. B. KNOBEL. Carnegie Institute of Washington, 1915, iv, 207 pp.+portrait. On account of various misleading statements made in histories, the following papers are of importance: J. L. E. DREYER, "On the origin of Ptolemy's catalogue of the stars," R. A. S., *Mo. Not.*, v. 77, 1917, pp. 528–539; v. 78, 1918, pp. 343–349. See also [92].

106a. Pages 1–34 of the following scholarly work is mainly devoted to discussion of PTOLEMY'S Geography: A. E. NORDENSKIÖLD, *Facsimile—Atlas to the Early History of Cartography with Reproductions of the Most Important Maps Printed in the XV and XVI Centuries. Translated from the*

Swedish Original by J. A. EKELÖF and C. R. MARKHAM. Stockholm, 1889, viii, 141 pp., 51 double map-plates. The first English translation of the Geography was published by the New York Public Library in 1932, in an edition of 250 copies and sold at $1000 each. See *Isis*, v. 20, 1933, pp. 270–274. On account of the text chosen for translation this edition was a most unfortunate one and highly unsatisfactory as a work of scholarly achievement.

106b. See ALBERT LEJEUNE, (a) "Les tables de réfraction de Ptolémée," Soc. Sci. de Bruxelles, *Annales*, s. 1, v. 60, 1946, pp. 93–101; (b) "Les lois de la réflexion dans l'optique de Ptolémée," *L'Antiquité Classique*, v. 15, 1947, pp. 241–256.

106c. A somewhat defective text of this work, together with a Latin translation by JOHN WALLIS was published at Oxford in 1680, and reprinted in his Collected Works, 1699. Compare F. J. FÉTIS, *Biographie Universelle des Musiciens . . .*, Paris, v. 7, 1870; also *The Oxford History of Music*, v. 1, by H. E. WOOLDRIDGE, Oxford, 1901, pp. 15–22; H. WYLDE, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, chap. XI, *etc.*; and D. B. MUNRO, *The Modes of Ancient Greek Music*, Oxford, 1894, pp. 108–112. Such works may also be consulted with regard to the musical work of EUCLID and PYTHAGORAS.

107. HEATH [22, 28], *History*, v. 2, pp. 355–459; *Manual*, pp. 434–465. Reference may also be given to J. H. WEAVER (a) "Pappus, Introductory Paper," Amer. Math. Soc., *Bull.*, v. 23, pp. 127–135, 1916; (b) "On foci of conics," *idem*, pp. 357–365, 1917; (c) "Some extensions of the work of PAPPUS and STEINER on tangent circles," *Amer. Math. Monthly*, v. 27, 1920, doctoral dissertation, Univ. of Pennsylvania, with "Life," 10 pp. This last deals with properties of the figure called "Shoemaker's Knife" studied by ARCHIMEDES and discussed by PAPPUS from whom we also learn of the interest of Archimedes in semi-regular polyhedra (see [83]). PAPPUS was also the author of a notable work: *The Commentary on Book X of Euclid's Elements. Arabic Text and Translation by William Thomson, with Introductory Remarks, Notes, and a Glossary of Technical Terms by* GUSTAV JUNGE & WILLIAM THOMSON. Cambridge, Harvard Univ. Press, 1930, 294 pp.

108. Of this work, originally in 8 books, Book 1 and the first 13 of 26 propositions in Book 2 are missing. There is a French translation of the Collection by PAUL VER EECKE. Paris and Bruges, 1933, cxxviii, 885 pp. There are also translations of two portions: (1) the latter part of Book 3, by J. S. MACKAY, "Pappus on the progressions," Edinb. Math. Soc., *Proc.*, v. 6, 1888, pp. 48–58; (2) a German translation of books 7–8 by C. I. GERHARDT, Halle, 1871, ii, 381 pp.

109. For example P. VER EECKE, "Le theorème dit de Guldin considéré au point de vue historique," *Mathesis*, v. 46, 1932, pp. 395–397. In *Scripta Mathematica*, v. 1, 1933, p. 268, it is shown that there is no foundation in fact for the argument of VER EECKE that GULDIN was an independent discoverer; rather does it confirm the belief that GULDIN had taken his result from the work of PAPPUS.

110. HEATH [22], *History*, v. 2, pp. 389–390, gives a translation of PAPPUS' "Preface on the sagacity of bees." See also the valuable survey, with references to literature, in D. W. THOMPSON, *On Growth and Form*, new edition. Cambridge, 1942, pp. 526–539. Also THOMAS [46a], v. 2, pp. 589–593.

111. We have been making numerous references to astronomy since it involved mathematics and, until long after the period we are considering, it was regarded as a part of mathematics. An old work of importance, almost wholly devoted to Greek astronomy, is J. B. J. DELAMBRE, *Histoire de l'Astronomie Ancienne*, Paris, 1817, v. 1, lxxii, 556 pp., 1 plate; v. 2, vii, 640 pp., 16 plates. The amount of presented detail, elaborated from source material, is especially valuable.

111a. THEON OF ALEXANDRIA, fourth century mathematician and astronomer, prepared editions of EUCLID's Elements and Optics [76], and with his daughter HYPATIA, mathematician and philosopher, prepared an elaborate commentary on PTOLEMY'S Almagest. There is a biography of HYPATIA in *Encycl. Britannica*, eleventh ed., v. 14, 1910. The story of her life was the basis of CHARLES KINGSLEY'S romance *Hypatia* (1853). Scholarly biographies of both Hypatia and her father are in PAULY-WISSOWA, *Real-Encyclopädie*, HYPATIA by K. PRAECHTER, v. 9, 1914, cols. 242–249; THEON by K. ZIEGLER, s. 2, v. 10, 1934, cols. 2075–2080. The following is the best text of *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste. Text établi et annoté par A.* ROME. Tome 1: *Pappus d'Alexandrie Commentaire sur les livres 5 et 6 de l'Almageste.* Tome II–III: *Théon*

d'Alexandrie, Commentaire sur les livres 1 et 2 et 3 et 4 de l'Almageste. 3 v. *Studie e Testi* 54, 72, 106. Rome, Vatican, 1931, 1936, 1943, paged continuously cxl, 1085 pp. See also [172].

111b. The Astrolabe is an instrument used not only for stellar but also for solar and lunar altitude observation. Its earliest forms may go back to the time of HIPPARCHUS; many varieties were developed by the Arabs and others up to the nineteenth century. The first Greek treatise on the astrolabe, which has been preserved, was by JOHN PHILOPONUS of Alexandria (*c.* 525). There is a French translation by P. TANNERY, "Jean le grammairien d'Alexandrie (Philopon), sur l'usage de l'astrolabe et sur les tracés qu'il présente," in his *Mémoires Scientifiques*, Toulouse and Paris, v. 9, 1929, pp. 341–367. A rather worthless English translation, by H. W. GREENE who evidently knew nothing of Greek scientific terminology, appears in v. 1, pp. 61–81 of R. T. GUNTHER, *The Astrolabes of the World*, 2 v., Oxford, 1932; this is a sumptuous and interesting work illustrated with over 150 plates. A recent excellent work discussing the history, theory and use of such instruments is HENRI MICHEL, *Traité de l'Astrolabe*, Paris, 1947, viii, 202 pp., with 24 plate reproductions. On pp. 164–165 are paragraphs about BÜRGI as a notable instrument maker. There are also references to the use of the linear astrolabe of NAṢĪR ED-DĪN AL-ṬŪSĪ (pp. 21, 115–122) and to the astrolabe and "Horizontall Dyall" of OUGHTRED (pp. 24, 129–130). The astrolabe was superseded by JOHN HADLEY'S much more accurate reflecting quadrant of 1731–34.

Of an earlier Greek work on the astrolabe instrument, by THEON of Alexandria, only the contents. in Syriac, have been preserved, but these are sufficient to indicate that the original was a main source for later writing. From PTOLEMY'S *Planesphaerium* (German translation by J. DRECKER in *Isis*, v. 9, 1927, pp. 255–278) it is suggested that HIPPARCHUS was familiar with stereographic projection, and it is clear that PTOLEMY knew of the construction and use of the plane astrolabe or Theon's "Little astrolabe." The use of the term astrolabe by PTOLEMY in the *Almagest* does not apply to this, but, as in the case of the *Hypotyposis* of PROCLUS, to the "armillary sphere."

The great fourteenth century poet GEOFFREY CHAUCER wrote in 1391 for his son, "Little Lowrys," a famous "Treatise on the Astrolabe. Bread and Milk for Children,"" of which an edition was published in R. T. GUNTHER, *Early Science in Oxford*, v. 5, Oxford, 1929. A better edition, perhaps, is that of W. W. SKEAT, London, 1872.

For many of the facts in this note I am indebted to Professor NEUGEBAUER, who has set forth the early history of the astrolabe in a memoir to be published in *Isis* in the near future. A reference may be given to MARCIA L. LATHAM, "The Astrolabe," *Amer. Math. Monthly*, v. 24, 1917, pp. 162–168, even though the distinction between astrolabe and "armillary sphere" has not been recognized.

112. P. K. HITTI, *History of the Arabs*, London, 1937, p. 166, gives information concerning Alexandrian libraries. "The great Ptolemaic Library was burnt as early as 48 B.C. by JULIUS CAESAR. A later one, referred to as the daughter Library, was destroyed about A.D. 389 at the command of Roman Emperor THEODOSIUS. At the time of the Arab conquest, therefore, no library of importance existed in Alexandria." A whole chapter is devoted to "The Library of Alexandria" in A. J. BUTLER, *The Arab Conquest of Egypt and the last Thirty Years of the Roman Dominion*, Oxford, 1902.

113. References may be given to the following recent publications dealing with Hindu mathematics before 600 A.D.:

B. DATTA, *The Science of the Śulba. A Study in Early Hindu Geometry*, Calcutta, 1932, xvi, 240 pp. The "śulbas" or "śulba-sūtras" are manuals for the construction of altars necessary in connection with sacrifices of vedic Hindus. While seven of these manuscripts, now available, are comparatively modern there is a wide range to the claims for antiquity of the originals—one authority placing 200 A.D. as an upper limit, another 500 B.C. The śulbas contain considerable discussion of mathematical interest, including a rule for squaring the circle. This rule, involving a segment of a circle and its apothem, suggests a possible connection with the problems showing that the Babylonians were familiar with the Pythagorean theorem about 2000 B.C. See also C. MÜLLER, "Die Mathematik der Śulvasûtra, eine Studie zur Geschichte indischer Mathematik," Hamburg Univ., Math. Sem., *Abhandlungen*, v. 7, 1929, pp. 173–204.

ĀRYABHATA (c. 476–c. 550) the elder was a noted Hindu astronomer whose celebrity rests on a work entitled *Āryabhaṭiya*, of which the second section, *Ganita* (66 lines) is devoted to mathematics

and is simply a set of rules which for the volume of a sphere is very inaccurate. It has been surmised that he attempted the general solution of a linear indeterminate equation. The following are references to literature: P. C. Sengupta, (a) "Āryabhaṭa, the father of Indian epicyclic astronomy," Univ. of Calcutta, *Jn. Dept. Letters*, v. 18, no. 3, 1929, 56 pp. (b) "The *Aryabhatiyam*, Translation," *idem*, v. 16, no. 6, 1927, 56 pp.; the *Ganita* with commentary occupies pp. 13–30. *The Āryabhaṭya of Āryabhaṭa. An Ancient Indian Work on Mathematics and Astronomy. Translated with notes* (and the use of SENGUPTA's works), by W. E. CLARK, Chicago, 1930, xxx, 90 pp. C. MÜLLER, "Volumen und Oberfläche der Kugel bei Āryabhaṭa I," *Deutsche Mathematik*, v. 5, 1940, pp. 244–255. S. GANGULI, "India's contribution to the theory of indeterminate equations of the first degree," Indian Math Soc., *Jn.*, v. 19, 1931, pp. 110–120, 129–142, 153–168; the first two parts deal with work of ĀRYABHAṬA the elder and the younger. Other references may be found in [114–115]. Had it not been for the lengthy review in *Math. Reviews*, v. 9, Feb. 1948, of the following miserable book, no reference to it would have been made here: L. V. GURJAR, *Ancient Indian Mathematics and Vedha*. Poona, 1947, vi, 202 pp. Another phase of its worthlessness is brought out in *Nature*, v. 161, Apr. 17, 1948, p. 580.

114. More general treatments of Hindu mathematics are to be found in the following: D. M. MEHTA, *Theory of Simple Continued Fractions (with special Reference to the History of Indian Mathematics)*. Diss. Heidelberg, Bhavnagar, India, 1931, iv, 164, 4 pp. G. CHAKRAVARTI, (a) "Growth and development of progressive series in India," (b) "The Hindu term for area," (c) "Surd in Hindu mathematics," (d) "On Hindu treatment of fractions," Univ. of Calcutta, *Jn. Dept. Letters*, v. 24, nos. 6–9, 76 pp. A. N. SINGH, "Use of series in Hindu mathematics," *Osiris*, v. 1, 1936, pp. 606–628.

115. Other general works of value are: G. R. KAYE, (a) *Indian Mathematics*. Calcutta and Simla, 1915, iv, 74 pp., 2 plates. (b) "Indian Mathematics," *Isis*, v. 2, 1919, pp. 326–356. B. DATTA & A. N. SINGH, *History of Hindu Mathematics. A Source Book. Part 1, Numerical Notation and Arithmetic*. 1935, xx, 261 pp. *Part 2, Algebra*. 1938, xvi, 314 pp. Lahore. Unreliable features of the first part are set forth in NEUGEBAUER's review in *Quellen u. Studien . . .*, v. 3B, pp. 263–271. See also DELAMBRE [111], v. 1, pp. 400–556.

116. H. G. ZEUTHEN, "Sur l'arithmétique géométrique des Grecs et des Indiens," *Bibl. Mathematica*, s. 3, v. 5, 1904, pp. 97–112.

117. Cajori [2], pp. 99–112; Smith [4] 1, pp. 164–177.

118. The great reference work for Arabian mathematicians, more than 500 of whom are listed, is H. SUTER, *Die Mathematiker und Astronomen der Araber und ihre Werke (Abhand. z. Gesch. d. mathem. Wissen.* Heft 10), Leipzig, 1900, x, 278 pp. (Also as Supplement to *Z. f. Math. u. Physik*, v. 45); "Nachträge und Berichtigungen," Heft 14, 1902, pp. 155–185. See also H. P. J. RENAUD, "Additions et corrections à Suter, 'Die Mathematiker . . . ,'" *Isis*, v. 18, pp. 166–183, 1932. Another Arabic work of first importance is H. SUTER, *Das Mathematiker-Verzeichnis in Firhrist des Ibn abt Ja'ḳûb an-Nadîm, zum erstenmal vollständig ins deutsche übersetzt und mit Anmerkungen versehen. (Abhand. z. Gesch. d. Mathem. Wissen.*, Heft 6), 1892, 87 pp. Also as Suppl. to *Z. f. Math. u. Physik*, v. 37. Ibn Ja'ḳûb an-Nadîm = Abu'l Farag Mah.b: Isháq.

119. We have already listed five works of this kind, [72, 81, 86, 103]. Among others were Arabic translations of works of APOLLONIUS, ARCHIMEDES, DIOPHANTUS, EUCLID, HERON, MENELAUS, PTOLEMY.

120. One of the main sources in English for the text of the chief mathematical work of BRAHMAGUPTA is *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhâscara, translated by* H. T. COLEBROOKE. London, 1817, lxxxiv, 378 pp. In the long introduction, pp. xxv–xxxvii deal mainly with BRAHMAGUPTA's astronomical work; pp. lxxx–lxxxiv contain a note on "communications of the Hindus with western nations on astrology and astronomy." BHĀSKARA's work on Arithmetic (Lilávatí) and Algebra (Vija-ganíta) fills pp. 1–276. BRAHMAGUPTA's Arithmetic (Ganíta) and Algebra (Pulverizer) occupies pp. 277–378.

121. In DATTA & SINGH [115], v. 2, pp. 204–245, are sections on rational triangles and rational quadrilaterals. See also DICKSON [101], pp. 191, 216; and L. E. DICKSON, "Rational triangles and quadrilaterals," *Amer. Math. Monthly*, v. 28, 1921, pp. 244–245.

122. It is said that Lilavati was the name of BHĀSKARA's daughter and that the book of that

name was written in her honor. The first English translation was *Bija Ganita: or the Algebra of the Hindus*. By E. STRACHEY, London, 1813, viii, 119 pp. There are also two revised editions of *Colebrooke's Translation of the Lîlâvatî with Notes*. By H. C. BANERJI, Calcutta, (a) 1893, viii, 174, 120 (Sanskrit text appendix) pp.; second ed., 1927, x, 202, 114 pp.

123. BHĀSKARA'S methods lead to $x=226153980$, $y=1766319049$ as the solution for the equation when $A=61$. Finding this result was proposed as a problem by FERMAT to FRÉNICLE DE BESSY in a letter of February 1647. EULER solved it in 1732. See DATTA and SINGH [115], v. 20, pp. 166–170. The cattle problem of ARCHIMEDES called for the solution of an equation of this type, with $A=4729494$; compare ARCHIBALD [80], or HEATH [22], *History*, v. 2, pp. 97–98. For details regarding tables of solutions of the equation $x^2-Ay^2=1$, $A\leqq 2000$ by DEGEN, CAYLEY, WHITFORD, D. H. LEHMER, see D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, Washington, 1941.

124. D. E. SMITH & L. C. KARPINSKI, *The Hindu-Arabic Numerals*. Boston, 1911, vi, 160 pp.

125. F. CAJORI, "The controversy on the origin of our numerals," *Scientific Mo.*, v. 9, 1919, pp. 458–464.

126. For many years the chief information in English regarding the first Arabic arithmetic, geometry, and algebra, was *The Algebra of Mohammed ben Musa edited and translated* by F. ROSEN. London, 1831, xvi, 208, 127 (Arabic text) pp. See S. GANDZ, "The sources of al-Khowârizmî's algebra," *Osiris*, v. 1, 1936, pp. 263–277. Dissatisfied with the ROSEN edition of the geometry section of his algebra, GANDZ brought out a new English edition, and showed that its rather trivial contents are mainly a reproduction of a Hebrew geometry of 150 A.D., displaying no evidence of Greek influence: *The Mishnat ha Middot the first Hebrew Geometry of about 150 C.E. and The Geometry of Muhammad ibn Musa al-Khowarizmi the first Arabic geometry (c. 820), representing the Arabic version of the Mishnat ha Middot. A new edition of the Hebrew and Arabic Texts with Introduction, Translation and Notes, Quellen u. Studien . . .*, v. 2A, 1932, pp. 61–85+certain parts of pages 86–96 (Literature and Index, etc.). The v. consists of 96 pp.+4 plates. GANDZ has also made elaborate correction of the "book on the legacies" in another section of ROSEN'S edition of al-Khowârizmî's Algebra, in "The algebra of inheritance, a rehabilitation of al-Khowarizmi," *Osiris*, v. 5, 1938, pp. 319–391. See also L. C. KARPINSKI, (a) "Robert of Chester's translation of the algebra of al-Khowarizmi," *Bibl. Math.*, s. 3, v. 11, pp. 125–131, 1911; (b) *Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, with an Introduction, Critical Notes and English Version by L. C. Karpinski. (Univ. of Michigan Studies, Humanistic Series*, v. 11). New York, 1915, viii, 164 pp.+4 plates. The Introduction occupies pp. 1–63. See also SARTON [8], v. 1, pp. 563–564; and a popular article, DUMONT, "Un professeur de mathématiques au IX^e siècle: Mohammed ibn Mousa Al-Khowarizmi," *Rev. gén. d. Sci. Pures et Appl.*, v. 54, no. 2, 1947, pp. 7–13.

127. SMITH & KARPINSKI [124], p. 52. See also B. DATTA, "Early literary evidence of the use of zero in India," *Amer. Math. Monthly*, v. 33, 1926, pp. 449–454.

128. F. CAJORI, "The zero and principle of local value used by the Maya of Central America," *Science*, n.s. v. 44, 1916, pp. 714–717.

129. GANDZ [21], pp. 509–541, has the following subsections: 11. "AL-KHOWÂRIZMÎ's algebra. Its character and its contributions"; 12. "The algebraic analysis of the Arabic types"; 13. "The geometric demonstrations, Al-K. and EUCLID"; 14. "The Babylonian problems in Al-K. algebra"; 15. "The analysis of type A II in Al-K. algebra"; 16. "Al-K. and DIOPHANTUS"; 17. "Al-K. and HERO." See also TROPFKE [98], pp. 95–99.

THUREAU-DANGIN, in "Un problème algébrique babylonien," *Halil Edhem Hatira Kitabi (Recueil offert à la Mémoire de Halil Edhem)*, Ankara, Türk Tarih Kurumu Basimeri, 1947, pp. 44–47, presents a Babylonian text dated about 1700 B.C. to show that of the three types of trinomial equations of the second degree distinguished by AL-KHOWÂRIZMÎ (1) $x^2+ax=b$, (2) $x^2+b=ax$, (3) $ax+b=x^2$, type 2 is also found in Babylonian mathematics. Al-K. employed no symbols but an algebraic geometry discussion. The Babylonians like Diophantus had numerical algebra. Types (1) and (3) were found there earlier (GANDZ [21], p. 142, and THUREAU-DANGIN [15], pp. xxi–xxiv, 1–10). See also especially THUREAU-DANGIN [34].

130. H. SUTER, *Die astronomischen Tafeln des Muhammed ibn Mûsâ al-Khowârizmî in der Bearbeitung des Maslama ibn Aḥmed al-Madjrîṭî unter der Latein. Uebersetzung des Athelhard von*

Bath auf Grund der Vorarbeitung von A. BJØRNBO *und* R. BESTHORN *herausgegeben und commentiert.* Copenhagen, 1914 (Danske Vidensk, Selsk., *Skrifter*, s. 7, *Hist. of Filos.*, v. 3, no. 1), Tables 58, 58a, 60. See also A. A. BJØRNBO, "Al-Chwârizmî's trigonometriske Tavler," pp. 1–17. of *Festskrift til* H. G. ZEUTHEN, Copenhagen, 1909.

131. SENGUPTA [113] (b) *Aryabhatiyam*, p. 11, a table of the values of 3438 sin A, with first differences for 24 angles $A = 3°45'(225')90°$. The values vary from 225 to 3438. See also A. VON BRAUNMÜHL, *Vorlesungen ü. Gesch. d. Trigon.*, 2 v. Leipzig, 1900, 1903; v. 1, p. 34.

132. H. SUTER, *Encyclopaedia of Islam*, v. 1, Leyden and London, 1908, pp. 112–113. SARTON [8], v. 1, pp. 666–667. J. B. J. DELAMBRE, *Histoire de l'Astronomie au Moyen Age*, 1819, pp. 156–170.

133. There are French and German translations of his geometry based on different manuscripts, as follows: F. WOEPCKE, "Analyse et extraits d'un receuil de constructions géométriques par Aboûl Wefâ," *Journal Asiatique*, s. 5, v. 5, 1855, pp. 218–256, 309–359, also as a reprint 89 pp. H. SUTER, "Das Buch der geometrischen Konstruktionen des Abu'l Wefâ" (*Abh. z. Gesch. d. Naturwiss. u. d. Medizin*, Heft 4). Erlangen, 1922, pp. 94–109. The history of various attempts at geometrical constructions with compasses with a single opening is given in W. M. KUTTA, "Zur Geschichte der Geometrie mit constanter Zirkelöffnung," K. Leop.-Carol. deutsch. Akad. Naturf., *Nova Acta*, v. 81, 1897, pp. 71–101+3 plates. See also L. RODET, *Bull. d. Bibl. d. Storia d. Sci. Matem. e Fis.*, v. 16, 1883, pp. 534–542.

134. Sin $30' = 0^{p}31'24''55'''54^{iv}55^{v} = .0087265373$, which should have been $0^{p}31'24''55'''-54^{iv}0^{v}17^{vi} = .0087265355$. BRAUNMÜHL [131], v. 1, p. 57, in this connection is decidedly erroneous. The value of sin 30′ was first given by RODET, in *Journal Asiatique*, s. 5, v. 15, 1860, p. 303.

135. W. E. STORY, *Omar Khayyâm as a Mathematician, read at a meeting of Omar Khayyâm Club of America, 6 April 1918.* Privately printed, Needham, Mass., Rosemary Press, 1919, 13 leaves +plates; also in *Twenty Years of the Omar Khayyâm Club of America*, 1921. Needham, Mass., leaves 70–72, 74, 75, 77, 78, 80, 81. G. SARTON, "The tomb of Omar Khayyâm," *Isis*, v. 29, 1938, pp. 15–19 with plate illustration of the magnificent tomb now at Nishâpûr; also J. FLEMING, "A pilgrim to Omar's forgotten tomb," *Travel*, v. 58, 1932, pp. 9–14. Among several imaginary sketches of the life of OMAR the following two may be mentioned: the attractively written H. MACFALL, *The Three Students*, London, 1926, viii, 351 pp.; and H. LAMB, *Omar Khayyam, a Life*. Garden City, N.Y., 1934, viii, 316 pp. The chief general English articles for scholars are in *Encyclopaedia of Islam*, v. 3, 1936, p. 985–989 by V. MINORSKY, and v. 1, 1912, p. 1006, art. "Djalâlî," by H. SUTER; and in SARTON [8], v. 1, pp. 759–761.

136. The famous English translation of this by EDWARD FITZGERALD first appeared in 1859 and was successively revised in 1868, 1872, 1879. A beautifully illustrated edition, "set forth in meter" by D. E. SMITH, and said to be more complete and more literal than FITZGERALD'S, was published at New York in 1933.

137. *L'Algèbre d'Omar Alkhayyâmî, publiée traduite et accompagnée d'extraits de manuscrits inédits* by F. WOEPCKE. Paris, 1851, xx, 128, 56 (Arabic text) pp.+5 folding plates. The following English translation by D. S. KASIR, professes to be a translation of an Arabic ms. in the D. E. SMITH Library of Columbia University: *The Algebra of Omar Khayyam.* Doctoral diss., Teachers' College, Columbia Univ., New York, 1931, vi, 126 pp. KASIR states that the Arabic ms. he used is practically identical with the one basic in WOEPCKE's work. See also TROPFKE [98], pp. 105–106. Special attention should be directed to a recent extensive paper concerning the binomial theorem and the extraction of cube and higher roots in Islamic mathematics: P. LUCKEY, "Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der islamischen Mathematik," *Mathem. Annalen*, v. 120, 1948, pp. 217–274. The names of OMAR KHAYYÂM and AL-BÎRÛNÎ, among many others, arise in the discussion.

138. SARTON [8], v. 2, pp. 1001–1013. R. STROTHMANN & T. RUSKA, *Encyclopaedia of Islam*, v. 4, 1934, pp. 980–982. SUTER [118], pp. 146–153.

139. *Traité du Quadrilatère attribué à Nassiruddin-el-Toussy . . . traduit par* A. CARATHÉODORY. Constantinople, 1891, ii, 214, 157 (Arabic text) pp. Review by (a) H. SUTER, *Bibl. Math.*, s. 2, v. 7, pp. 1–8, the best; (b) P. TANNERY, *Bull. Sci. Math.*, s. 2, v. 16, 1892, pp. 147–152; (c) CARRA DE VAUX, *Jn. Asiatique*, s. 8, v. 20, 1892, pp. 176–181. See also Sister MARY CLAUDIA ZELLER, *The Development of Trigonometry from Regiomontanus to Pitiscus*, Diss. Univ. Mich., 1946, pp. 9–13.

A. VON BRAUNMÜHL, "Nassîr Eddîn Tûsi und Regiomontan," K. Leopold-Carol. deutsch. Akad. d. Naturf., *Nova Acta*, v. 81, 1897, pp. 39–68+2 plates. In his discussion of celestial spheres NASÎR ED-DÎN notes incidentally that the locus of a point on the circumference of a circle C_1 rolling inside the circumference C_2, whose radius is twice that of C_1 is a diameter of C_2; see P. TANNERY, *Recherches sur l'Histoire de l'Astronomie Ancienne*. Paris, 1893, viii, 370 pp. p. 348.

140. D. E. SMITH, "Euclid, Omar Khayyâm, and Saccheri," *Scripta Mathematica*, v. 3, 1935, pp. 5–10. In order to have here a true picture of what is known, the following facts should also be borne in mind. At least three editions of NASÎR ED-DÎN'S *Euclidis elementorum libri XII studii Nassiredini*, an Arabic work with Latin title, were published (1594, 1657, 1801). The part of it dealing with EUCLID'S fifth postulate was translated into Latin, in JOHN WALLIS, *Opera*, v. 2, 1693, pp. 669–673. See HEATH [73], v. 1, pp. 208–210; but especially BONOLA [78], where it is made clear that SACCHERI'S work on noneuclidean geometry was started by his knowledge of NASÎR ED-DÎN'S writings. If scholars later agree that in this discussion NASÎR ED-DÎN is reporting on a work of OMAR KHAYYÀM, it would certainly be a matter of exceptional interest.

141. L. BOUVAT, *Encycl. of Islam*, v. 4, 1934, pp. 994–996; DELAMBRE [132], pp. 204–211. *Ulugh Beg's Catalogue of Stars revised from all Persian Manuscripts existing in Great Britain . . .* by E. B. KNOBEL, as published by the Carnegie Institution of Washington in 1917, 109 pp., is the best. There is information about ULUGH BEG in this introduction. This Catalogue was made mainly from original observations. See also DELAMBRE [132], pp. 204–211.

142. AL-BÎRÛNÎ-SCHOY [81], pp. 92–108. Tables of sin A, $A=0(1')44°59'$, and of tan A, $A=0(1')45°$ are given here; the latter table was continued at 5′ interval to 90°, and of sin A, at interval 1′. See also C. SCHOY, "Beiträge zur arabischen Trigonometrie," *Isis*, v. 4, 1923, pp. 398–399, where 7 values of sin A, from $A=57°4'$ to $A=89°59'$ are given. This article of SCHOY is otherwise of great interest and value.

143. See BRAUNMÜHL [131], v. 1, pp. 72–75. If the radius $=60^p$ the cubic equation is $2700^p x = x^3+900^p$ sin 3°, whence $x=\sin 1°=1^p2'49''43'''11^{IV}$ is found. The more accurate value is $\sin 1°=1^p2'49''43'''11^{IV}14^V44^{VI}=0.017\,452\,406\,437$. It seems probable that the Arabs, long before the time of ULUGH BEG, knew how to find sin 1°. See WOEPCKE, "Discussion de deux méthodes Arabes pour déterminer une valeur approchée de sin 1°," *Jn. d. Math. Pures et Appl.*, v. 19, 1854, pp. 153–176; and also WOEPCKE [137], p. 125. If in Ulugh Beg's cubic we set $r=1$, we get the well-known relation $\sin 3°=3\sin 1°-4\sin^3 1°$.

144. CANTOR [12], v. 2, pp. 1–53; SARTON [8], v. 2, pp. 611–613; ARCHIBALD [72]; TROPFKE [98]; *Gli Scienziati Italiani*, v. 1, part 1, Rome, 1921, "Leonardo Fibonacci," by G. LORIA, pp. 4–12, excellent sketch of life and works, with bibliography. Most of LEONARDO'S writings have been brought together in two very large volumes, edited by Prince BALDASSARE BONCOMPAGNI: *Scritti di Leonardo Pisano*, Rome; v. 1, 1857, 459 pp., contains the *Liber Abaci*; v. 2, 1862, with *Practica Geometriae*, pp. 1–224; *Flos*, pp. 227–247; *Liber Quadratorum*, pp. 253–283.

145. See [144] and CANTOR [12], v. 2, pp. 5–35; K. VOGEL, "Zur Geschichte der linearen Gleichungen mit mehreren Unbekannten," *Deutsche Mathematik*, v. 5, 1940, pp. 217–240. This paper deals with 29 problems on pp. 228–258 of section 12 (extending from pp. 166 to 318) in the 15 sections of *Liber Abaci*. The contents of the other sections are suggested by the following headings: 1. Reading and writing of numbers in the Hindu-Arabic system; 2. Multiplication of integers; 3. Addition of integers; 4. Subtraction of integers; 5. Division of integers; 6. Multiplication of integers by fractions; 7. Further work with fractions; 8. Prices of goods; 9. Barter; 10. Partnership; 11. Alligation; 13. Rule of false position; 14. Square and cube roots; 15. Geometry and algebra, the former devoted to mensuration.

146. D. E. SMITH, "On the origin of certain typical problems," *Amer. Math. Monthly*, v. 24, 1917, pp. 64–71.

147. If u_n denotes the nth term of this series, DANIEL BERNOULLI in 1724, and EULER in 1726, found that

$$u_n=\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]:\sqrt{5}.$$

The literature connected with this series is very extended, and in different directions, such as

golden section, decomposition of large numbers, mathematical puzzles, and older discussions of leaf arrangement or phyllotaxis. Some suggestions in this connection may be found in R. C. ARCHIBALD, (a) "Golden section" and "Fibonacci series," *Amer. Math. Monthly*, v. 25, 1918, pp. 232–238; (b) "Notes on the logarithmic spiral, golden section and the Fibonacci series," pp. 146–157 of J. HAMBIDGE, *Dynamic Symmetry*, New Haven, 1920. See also D. W. THOMPSON, *On Growth and Form*, second ed., Cambridge, 1942, pp. 912–933; L. E. DICKSON, *History of the Theory of Numbers*, Washington, D.C., v. 1, 1919, pp. 393–411; and D. YARDEN, "A bibliography of the Fibonacci sequence," *Riveon Lematematika*, v. 2, Jan. 1948, pp. 36–45, covering the period 1202–1947.

148. R. B. MCCLENON, "Leonardo of Pisa and his Liber Quadratorum," *Amer. Math. Monthly*, v. 26, 1919, pp. 1–8.

149. *Flos, Scritti*, p. 234 [**144**]; $x = 1^{p}22'7''42'''33^{IV}4^{V}40^{VI} = 1.36880\ 81078\ 532$ instead of 1.36880 81078 213. But more remarkable than the approximation of this root is LEONARDO's discussion of the equation in attempting to prove that a geometrical construction of a root with ruler and compasses was impossible. As BELL [5] remarks, there was nothing in algebra like the inspiration for this attempted proof until the nineteenth century. See J. P. GRAM, "Essai sur la restitution du calcul de Leonard de Pise sur l'équation $x^3+2x^2+10x=20$," Danske Vidensk. Selskabs, *Meddelelser*, no. 1, 1893, pp. 18–28; F. WOEPCKE, "Sur un essai de déterminer la nature de la racine d'une équation du troisième degré . . . ," *Jn. d. Math. Pures et Appl.*, v. 19, 1854, pp. 401–406; A. GENOCCHI, *Annali di Scienze Matem., Fisiche*, v. 6, 1855, pp. 161–168; Q. VETTER, (a) "Nota alla risoluzione dell' equazione cubica di Leonardo Pisano," R. Accad. d. Sci. d. Torino, *Atti, Cl. d. Sci. Fis., Matem. e Nat.*, v. 63, 1928, pp. 296–299; (b) also in Czechish, *Časopis pro pěstování Matem. a Fysiky*, v. 58, 1928, pp. 149–151. A solution is obtained by the method of false position. To illustrate Arabic solutions of cubics, similar to Leonardo's, but two centuries earlier, see AL-BÊRÛNÎ-SCHOY [**81**], pp. 19, 21, where for the equation $x^3=1+3x$ the root is given as $1^{p}52'45''47'''13^{IV} = 1.879\ 385\ 246\ 8$ whereas the more accurate approximation is 1.879 385 241 8; for $x^3+1=3x$, $x=0^{p}20'50''16'''1^{IV}=0.347\ 296\ 373\ 5$, where the better value is 0.347 296 355 3. These equations come up in AL-BÊRÛNÎ's discussion of the construction of a regular polygon of nine sides. For Greek solutions of cubic equations see THOMAS [**64a**], v. 2, pp. 133, 137–159, 163, 539–541.

150. J. G. HAGEN, *The Catholic Encyclopedia*, v. 10, New York, 1911, pp. 628–629; A. M. CLERKE, *Encycl. Britannica*, eleventh ed., v. 23; Delambre [**132**], pp. 288–365; CANTOR [**12**], v. 2, pp. 252–289; A. ZIEGLER, *Regiomontanus (Joh. Müller aus Königsberg in Franken) ein geistiger Vorläufer des Columbus*. Dresden, 1874, iv, 104 pp. Astronomical Ephemerides (1473–74) of REGIOMONTANUS with positions of the sun, moon and planets and the eclipses 1475 to 1506, guided COLUMBUS to America. See also pp. 304–305, 320–323 of J. D. BOND, "The development of trigonometrical methods down to the close of the XVth century (with a general account of the methods of constructing tables of natural sines down to our days)," *Isis*, v. 4, pp. 295–323, 1922. Also Sister M. C. ZELLER [**139**] pp. 17–36; *A Source Book in Mathematics*, ed. by D. E. SMITH, New York, 1929, pp. 427–433, "On the law of sines for spherical triangles" and "On the relations of the parts of a triangle."

151. In an essentially astrological work, *Tabulae Directionum Professionumque* (1490) REGIOMONTANUS gives a table of tangents for each degree of the quadrant of a circle of radius 100 000. Up to 45° the error in the last figure does not exceed 2, but later, the error is much larger.

152. D. E. SMITH, "The first printed arithmetic (Treviso, 1478)," *Isis*, v. 6, 1924, pp. 311–331. See also D. E. SMITH, *Rara Arithmetica*, Boston, 1908; *Addenda*, 1939.

153. For various printed editions of Euclid's *Elements*, see P. RICCARDI, *Saggio di una Bibliografia Euclidea*, 5 parts, Bologna, 1887–1893; C. THOMAS-STANFORD, *Early Editions of Euclid's Elements*. London, 1926, vi, 67 pp.+13 pls.

154. CANTOR, v. 2, pp. 306–344; H. STAIGMÜLLER, "Lucas Paciuolo. Eine biographische Skizze," *Z. f. Math. u. Physik, Hist. Abt.*, v. 34, 1889, pp. 81–102, 121–128. Of PACIOLI's *Sūma* there was a second edition in 1523 and of the divine proportion there was a German edition with commentary by C. WINTERBERG (Vienna, 1896, iv, 365 pp.). See also S. MORISON, *Fra Luca de Pacioli of Borgo S. Sepolcro*. New York, The Grolier Club, 1933, x, 106 pp.

155. J. B. GEIJSBEEK, *Ancient Double-Entry Book-keeping Lucas Pacioli's Treatise (A.D.*

1494—The earliest known writer on bookkeeping) *reproduced and translated with reproductions, notes and abstracts from Manzoni, Pietra, Mainardi, Ympyn, Stevin, and Dafforne.* Denver, Colo., 1914, iv, 182 pp.

156. In the main I have here followed CANTOR [12], v. 2, pp. 482–496, and H. WIELEITNER, "Über Cardan's Beweis für die Lösung der kubischen Gleichung," Physik.-medizinische Sozietät, *Sitzb.*, Erlangen, v. 58–59, 1928, pp. 173–176. Another point of view is set forth by TROPFKE [11], v. 3, p. 136–139; perhaps his is the more correct.

157. J. CARDAN, *The Book of My Life* (*De Vita Propria Liber*) translated from the Latin by Jean STONER, New York, 1930, xx, 331 pp.; SMITH [54]; R. B. LINDSAY, "Jerome Cardan, 1501–1576," *Amer. Jn. Physics*, v. 16, May, 1948, pp. 311–317; W. G. WALTERS, *Jerome Cardan, a Biographical Study.* London, 1898, vi, 301 pp.; H. MORLEY, *The Life of Girolamo Cardano, of Milan, Physician.* 2 v. London, 1854, xii, 304, iv, 308 pp.; S. GÜNTHER, *Allgemeine Deutsche Biographie*, v. 28, 1889, pp. 388–390; TROPFKE [98]; D. E. SMITH, "Medicine and mathematics in the sixteenth century," *Annals of Medical History*, 1917, p. 130. Of the 528 mathematicians and astronomers listed in SUTER [118], at least 87 are known to have been physicians; see also [162]. ARCHIBALD [57] tells about Cardan's work on music. JAMES ECKMAN, *Jerome Cardan.* (*Supplements* to the *Bulletin of the History of Medicine*, no. 7). Baltimore, 1946, xiv, 120 pp. Under "References" are 335 titles.

158. An English translation, with commentary, of the solutions of cubic and quartic equations occurring in CARDAN'S work, is given by R. B. MCCLENON in SMITH [150], pp. 203–212. CARDAN gave the modern transformation for reducing a four-term equation of the third degree to one lacking the second-degree term; he recognized that a cubic equation had three roots, which might be negative or irrational, and he knew that the sum of the roots of a cubic equation was equal to the coefficient of the term of the second degree (see TROPFKE [11], v. 3, pp. 138–139).

159. For some details in this regard see J. W. L. GLAISHER, "Report on mathematical tables," BAAS *Report 1873*, pp. 43–45, 158; A. DEMORGAN, (a) art. "Table" in *The English Cyclopaedia, Arts and Science Section*, v. 7, 1861, cols. 984, 988; (b) "On the almost total disappearance of the earliest trigonometrical canon," RAS *Mo. Not.*, v. 6, 1845, pp. 221–228; reprinted with a small addition in *Phil. Mag.*, s. 3, v. 26, 1845, pp. 517–526. This last article is of exceptional interest and value. See also DELAMBRE [160], v. 2, pp. 1–35. In a 22-page *Canon Doctrinae Triangulorum*, Leipzig, 1551, RHETICUS has a 7-place table of the trigonometric functions at interval 10′.

160. K. LUNDMARK, "Nicolaus Kopernikus (Kopernik) and his astronomical reformation, introductory observations," Lund, Universitet, Observatoriet, *Meddelanden*, s. 2, no. 112, 1944, 18 pp. (*Historical Notes and Papers*, no. 19). A. ARMITAGE, *Copernicus, the Founder of Modern Astronomy.* London, 1938, ii, 183 pp. *Nicholas Copernicus, a Tribute of Nations.* Ed. by S. P. MIZWA. New York, 1945, xix, 268 pp. O. J. LODGE, *Pioneers of Science*, London, 1919, "Copernicus, and the motion of the earth," pp. 3–31. DREYER [89], pp. 305–344. SMITH [54]. L. C. KARPINSKI, "The progress of the Copernican theory," *Scripta Math.*, v. 9, 1943, pp. 139–154, illustr. DELAMBRE, *Histoire de l'Astronomie Moderne*, 2 v., Paris, 1821, v. 1, pp. 85–142. My reasons for calling Copernicus a Polish astronomer are that he was born in Thorn, Poland; that his father was born in Cracow, Poland; that his mother's brother was a bishop of Ermeland, Poland. Ermeland was Polish 1466–1772, and Thorn 1466–1793.

161. There was a German translation of the *De Revolutionibus* in 1879, a French translation in 1934, and an English translation by C. G. WALLIS, *On the Revolution of the Celestial Spheres*, 1939. This last is not, however, available for general distribution, but is for consultation in the Library of St. John's College, Annapolis, Md. There is available, however, *Nicolaus Copernicus, De Revolutionibus Preface and Book I translated by* J. F. DOBSON & S. BRODETSKY, *with a Biographical Note and Notes on the Translation.* RAS, *Occasional Notes*, no. 10, May 1947, ii, 32 pp., 2 plates. Even though it is not easy for a library to procure this publication, it is to be hoped that this translation, although somewhat abridged, may be continued. In order to prevent useless search the following notes may be added: In *Nature*, v. 106, p. 515, 16 Dec. 1920, a note states that DOBSON & BRODETSKY had nearly completed their translation of *De Revolutionibus.* This note was elaborated by F. E. BRASCH in *Science*, n.s., v. 64, 13 Aug. 1926, stating that printers' proof of the trans-

lation was being read but that the v. was to be published by the Oxford Univ. Press. The v. has not yet appeared. Many years ago I saw the original manuscript of *De Revolutionibus* in the City Hall at Prague. Although this Hall was destroyed during the recent war, I learn that the manuscript is still intact in Prague. See also G. McCOLLEY, "The universe of *De Revolutionibus*," *Isis*, v. 30, pp. 452–472. An admirable introduction to ideas of COPERNICUS is given by E. ROSEN in *Three Copernican Treatises: The Commentariolus of Copernicus, the Letter against Werner, the Narratio Prima of Rheticus; translated with Introduction and Notes*. New York, 1939, x, 211 pp.

162. ROBERT RECORDE (1542) was the author of four books on mathematics and one on medicine, for he was also physician to King EDWARD VI and Queen MARY. There were many editions of his *Ground of Arts* (1541?), arithmetic; his *Castle of Knowledge* (1551), astronomy, dealt with the Copernican System; the *Whetstone of Witte* (1557), algebra, was the source of our sign for equality; L. C. KARPINSKI, "The Whetstone of Witte (1557)", *Bibliotheca Math.*, s. 3, v. 13, 1913, pp. 223–238. See SMITH [4], v. 1, pp. 317–321. Also D. E. SMITH, "New information respecting Robert Recorde," *Amer. Math. Monthly*, v. 28, 1921, pp. 296–300; F. V. MORLEY, "Finis coronat opus," *Scientific Monthly*, v. 10, 1920, pp. 306–308; F. M. CLARKE, "New light on Robert Recorde," *Isis*, v. 8, 1926, pp. 50–70; F. R. JOHNSON and S. V. LARKEY, "Robert Recorde's mathematical teaching and the anti-Aristotelian movement," *The Huntington Library Bull.*, no. 7, 1935, pp. 59–87. A work including quaintly expressed biographical notes concerning seventeenth century mathematicians is: "*Brief Lives*," *Chiefly of Contemporaries, set down by* JOHN AUBREY *between the Years 1669 & 1696. Edited from the Author's MSS.* by ANDREW CLARK. 2 v. Oxford, 1898, xvi, 428. iv, 372 pp., 8 plates.

163. Admirable surveys of the life and work of STEVIN and of the history of decimal fractions have been given by G. SARTON, (a) "Simon Stevin of Bruges (1548–1620)," *Isis*, v. 21, 1934, pp. 214–303+2 plates; (b) "The first explanation of decimal fractions and measures (1585). Together with a history of the decimal idea and a facsimile (no. XVII) of STEVIN's Disme," *Isis*, v. 23, 1935, pp. 153–244. Then there's the recent fine volume, in Dutch, *Simon Stevin* by E. J. DIJKSTERHUIS. The Hague, 1943, x, 379 pp., 7 plates. He had earlier published "Simon Stevin und seine Bedeutung für die Geschichte der Mathematik und Naturwissenschaften," *Unterrichtsblätter f. Mathem. u. Naturw.*, v. 38, 1932, pp. 148–150. R. C. ARCHIBALD, *Mathematical Table Makers. Portraits, Paintings, Busts, Monuments, Bio-Bibliographical Notes*. New York, 1948, pp. 74–76, 82. See also F. CAJORI, *A History of Mathematical Notations*, 2 v., Chicago, 1928–1929, v. 1, pp. 154–158, 314–315, etc.

164. R. C. ARCHIBALD, *Math. Tables and Other Aids to Computation*, v. 1, pp. 400–402, 1945; v. 2, pp. 91–92, 1946.

165. A good account of VIETA and his works, with other material, is given by A. DEMORGAN in the *Penny Cyclopaedia*, v. 26, 1843, pp. 311–317; and also in *The English Cycl.-Biog.*, v. 6, 1858, cols. 361–371. See also C. HUTTON, *A Philosophical and Mathematical Dictionary*, new ed., v. 2, London, 1815; J. L. F. BERTRAND, *Éloges Académiques*, n.s., Paris, 1902, "La vie d'un savant au xvi siècle," pp. 143–146; ARCHIBALD [163], pp. 81–82; SMITH [54]; and CAJORI [163], v. 1–2.

166. Q. VETTER, "Sur l'équation du quarante-cinquième degré d'Adriaan van Roomen," *Bull. d. Sciences Math.*, s. 2, v. 54, 1930, pp. 277–283; and BRAUNMÜHL [131], v. 1, pp. 169–170. DELAMBRE [132], pp. 455–483; C. HUTTON, *Mathematical Tables*, sixth ed., London, 1822, "History of trigonometrial tables," pp. 4–9. These last two items tell something of VIETA's remarkable tables of trigonometric functions, *Canon Mathematicus seu ad Triangula*, 1679. See also DEMORGAN (a) [159], cols. 985–986 and K. HUNRATH, "Des Rheticus Canon doctrinae triangulorum und VIETA's Canon Mathematicus," *Abh. z. Gesch. d. Math.*, v. 9, 1899, pp. 211–240.

167. In a publication of 1593 VIETA derived an expression for the area of a unit circle, which was equivalent to finding the following relation:

$$2/\pi = \sqrt{\tfrac{1}{2}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots.$$

(The unconditional convergence of this product was proved by F. RUDIO, *Z. Math. Phys.*, v. 26, 1891, *hist.-lit. Abt.*, pp. 139–140.) This is a particular case of the relation obtained by EULER in 1737

$$\sin\theta/\theta = \cos\tfrac{1}{2}\theta\cdot\cos\tfrac{1}{4}\theta\cdot\cos\tfrac{1}{8}\theta\cdots, \quad \theta < \pi.$$

168. *The Sumario Compendioso of Brother Juan Diez, the Earliest Mathematical Work of the New World.* Boston, 1921, vi, 65 pp., frontispiece, is a little work edited by D. E. SMITH. It gives a facsimile reproduction and translation of pages of chief mathematical interest, with introductory material. See also D. E. SMITH, "The first work on mathematics printed in the New World," *Amer. Math. Monthly*, v. 26, 1921, pp. 10–15. L. C. KARPINSKI, *Bibliography of Mathematical Works Printed in America through 1850*, Ann Arbor, Mich., 1940, p. 25, lists this work by "Juan Diez freyle" (Brother Juan Diez) as by Freyle, Juan Diez. The earliest arithmetic published in America was that of PEDRO DE PAZ at Mexico City in 1623; see F. CAJORI, *Isis*, v. 9, 1927, pp. 391–401+1 plate.

It is a striking fact that in the Old World not until the nineteenth century (1825) was the first book published in Athens, Greece.

169. W. R. MACDONALD, *Dict. Nat. Biog.*, v. 40, 1894, pp. 59–65. J. W. L. GLAISHER, "Napier, John," *Encycl. Britannica*, eleventh ed., v. 19, 1911, pp. 171–175; also "Logarithm," v. 16, 1911, pp. 868–877. ARCHIBALD [163], pp. 58–63. *Napier Tercentenary Memorial Volume*, ed. by C. G. KNOTT. Publ. by the R. Soc. Edinburgh, London, 1915. xii, 442 pp. [Partial contents: LORD MOULTON, "The invention of logarithms, its genesis, and growth," pp. 1–32+6 pp. of plates; P. H. BROWN, "John Napier of Merchiston," pp. 33–51; F. CAJORI, "Algebra in Napier's day and alleged prior inventions of logarithms," pp. 93–109; D. M. Y. SOMMERVILLE, "Napier's rules and trigonometrically equivalent polygons," pp. 169–176]. *Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition*, edited by E. M. HORSBURGH, London, 1914, viii, 343 pp.+plates; G. A. GIBSON, "Napier and the invention of logarithms," pp. 1–16. SMITH [54]. The word logarithm, employed by NAPIER in 1614, is found much earlier in a 1553 work on divination by CASPAR PEUCER; "among the kinds of arithmanteia there is one we call by the new name logarithmanteia." For further details see F. CAJORI, *Archeion*, v. 12, 1930, pp. 229–233.

170. J. GINSBURG, "The Napier bones," SMITH [150], pp. 182–185; W. D. CAIRNS, "On the table of logarithms," SMITH [150], pp. 149–155.

171. E. W. HOBSON, *John Napier and the Invention of Logarithms, 1914*, Cambridge, 1914, 48 pp. H. S. CARSLAW, "The discovery of logarithms by Napier," *Math. Gazette*, v. 8, 1915, pp. 76–84, 115–119. R. C. ARCHIBALD, "Napier's *Descriptio* and *Constructio*," Amer. Math. Soc., *Bull.*, v. 22, pp. 182–187, 1916. *The Construction of the Wonderful Canon of Logarithms by* JOHN NAPIER. *Translated from Latin into English with Notes and a Catalogue of the Various Editions of Napier's Work by* W. R. MACDONALD. Edinburgh and London, 1889, xx, 169 pp.

172. TROPFKE [11], v. 2, pp. 217–218. The only rival of NAPIER in the invention of logarithms was the Swiss JOOST BÜRGI, who published a table of logarithms *Arithmetische und Geometrische Progress Tabulen*, at Prague in 1620. This table was conceived and constructed independently of NAPIER. The base of these logarithms was $(1.001)^{-1}$. For sources of information about BÜRGI see ARCHIBALD [163], pp. 15, 82. See also CAJORI [169] and KNOTT [169], R. A. SAMPSON, "The discovery of logarithms by Jobst Buergi," pp. 208–218+plate. Also L. DEFOSSEZ, *Les Savants du XVIII^e S. et la Mesure du Temps*. Lausanne, 1946.

173. The gap from 20000 to 90000, to ten places of decimals, was filled in by DEDECKER & ADRIAEN VLACQ in a work published at Gouda, Holland in 1628 (English edition, London, 1631). It was a comparatively easy matter to fill this gap to 10 places from what BRIGGS had already done in his *Arithmetica Logarithmica* to 14. Rushing into print with this as a second edition of the work of BRIGGS, at a time when it was known that BRIGGS had nearly completed his table to 14 places is not easily condoned. For a biography of BRIGGS by THOMAS WHITTAKER, see *Dict. Nat. Biog.* v. 6, 1886, pp. 326–327. On pp. 10–11 of his 1624 volume BRIGGS found the value of $\log_{10} 1.(000)^{51} = 0.(0)^{16}43429\ 44819\ 03251\ 804$. Since $\log_{10} e = .43429\ 44819\ 03251\ 82765\ \ldots$, and $e \approx (1+10^{-14})^{10^{14}}$, Briggs really gave $\log_{10} e$ correct to 16D.

174. *Logarithmica Britannica, being a Standard Table of Logarithms to Twenty Decimal Places.* By A. J. THOMPSON, Cambridge, Engl., Parts 1–10, 1924–1949, each part containing the logarithm of 10000 numbers. In several of the parts are some interesting facsimile reproductions, such as the will of HENRY BRIGGS, the title-pages of his work of 1624, the title and specimen pages of his tract of 1617, *Logarithmorum Chilias Prima*, and pages illustrating the relation of BRIGGS to NAPIER'S *Constructio Canonis*, 1619. See also ARCHIBALD [163], "A. J. Thompson," pp. 78–79 and portrait.

175. See E. BORTOLOTTI, *L'École Mathématique de Bologne, Aperçu historique*. Bologna, 1928,

pp. 35–38; among mathematicians at Bologna before 1800 were the following: PACIOLI, TARTAGLIA, CARDAN, FERRARI, CAVALIERI, RICCATI, AGNESI. L. BRUNSCHVICG, *Les Étapes de la Philosophie Mathématique*. Paris, 1912, pp. 162–167.

176. *Johann Kepler 1571–1630, A Tercentenary Commemoration of his Work . . . prepared under the Auspices of the History of Science Society*. Baltimore, 1931. [Contents: W. C. RUFUS, "Kepler as an astronomer," pp. 1–38; D. J. STRUIK, "Kepler as a mathematician," pp. 39–57 (admirable sketch); F. E. BRASCH, "Bibliography of Kepler's Works," pp. 86–133]. The best Kepler bibliography, with many facsimile reproductions of title pages, is *Bibliographia Kepleriana. Ein Führer durch das gedruckte Schriftum von Johannes Kepler mit 80 Faksimile*. By M. CASPAR, Munich, 1936, 160, 86 pp. Many other references to material about KEPLER including a discussion of numerous false portraits are to be found in ARCHIBALD [163], pp. 34–42. DREYER [89], pp. 372–412. There are exactly 9 regular solids: the 5 convex solids of the ancient Greeks, and the four star polyhedra, two discovered by KEPLER; see H. S. M. COXETER, "The nine regular solids," *Proc. of the First Canadian Math. Congress Montreal 1945*, Toronto, 1946, pp. 252–264. The best biography of BRAHE is J. L. E. DREYER, *Tycho Brahe, A Picture of Scientific Life and Work in the Sixteenth Century*. Edinburgh, 1890, xvi, 405 pp. T. BRAHE, *Opera Omnia*, Copenhagen, 15 v. 1913–1929.

177. HART [60], pp. 102–124. J. J. FAHIE, (a) *Galileo his Life and Work*. New York, 1903, xvi, 401 pp. *Memorials of Galileo, 1564–1642. Portraits and Paintings, Medals, Medallions, Busts, and Statues, Monuments, Mural Inscriptions*. London, 1929. (b) "The scientific work of Galileo (1564–1642) with some account of his life and trial," *Studies in the History of Science*, ed. by C. SINGER, v. 2, Oxford, 1921, pp. 206–284+3 plates; this is a most admirable revealing of Galileo's greatness as a scientist. An excellent brief summary is to be found in chapter 6, pp. 75–95 of H. F. GIRVIN, *A Historical Appraisal of Mechanics*, Scranton, Pa., 1948, xii, 275 pp. A. KOYRÉ, "Galileo and the scientific revolution of the seventeenth century," and L. OLSCHKI, "Galileo's philosophy of science," *Philosophical Review*, v. 52, 1943, pp. 333–348 and 349–365. *Portraits of Famous Physicists, with Biographical Accounts* by HENRY CREW, New York, 1942. [An excellently edited portfolio: GALILEO, HUYGENS, NEWTON, AMPÈRE, FRESNEL, FARADAY, JOULE, CLAUSIUS, MAXWELL, GIBBS, HERTZ, ROWLAND].

LANE COOPER, *Aristotle, Galileo, and the Tower of Pisa*. Ithaca, N. Y., 1935. R. T. GUNTHER, "Galileo and the leaning tower of Pisa," *Nature*, v. 136, 1935, pp. 6–7 (review). A. S. EVE, "Galileo and scientific history. The leaning tower and other stories," *Nature*, v. 137, 1937, pp. 8–10. That GALILEO really experimented with falling bodies at the leaning tower is here made to appear somewhat uncertain, so far as evidence available is concerned.

178. The bridge, by ESSEX, across the Cam in the grounds of Trinity College, Cambridge, has cycloidal arches (BALL [1], p. 287).

179. A. M. CLERKE, *Dict. Nat. Biog.*, v. 24, 1890, pp. 437–439. H. STEVENS, *Thomas Hariot the Mathematician, the Philosopher and the Scholar . . .*, London, Privately printed, 1900. Also F. V. MORLEY, "Thomas Hariot—1560–1621", *Scientific Monthly*, v. 14, 1922, pp. 59–65; F. CAJORI, "A revaluation of Harriot's *Artis Analyticae Praxis*," *Isis*, v. 11, 1928, pp. 316–324; and *Supplement to Dr. [James] Bradley's Miscellaneous Works with an Account of Harriot's Astronomical Papers*, Oxford, 1833, pp. 17–70+5 plates with facsimiles of Harriot manuscripts. Anyone desiring carefully to study HARRIOT'S *Artis Analyticae Praxis* would, in view of conflicting history statements, naturally read the survey in J. WALLIS, *A Treatise of Algebra*. Oxford, 1685, pp. 126–200, before turning to CAJORI'S "Revaluation," referred to above. The first paragraph (p. 126) is as follows: "Mr. *Harriot* in his Postumous Treatise of *Algebra* or *Analytica*, (published by Mr. *Walter Warner*, in the year 1631; soon after the first edition of Mr. *Oughtred's Clavis*, in the same year;) doth in divers things vary from the Method of *Vieta* and *Oughtred*. And hath made very many advantagious improvements in this art; and hath laid the foundation on which *Des Cartes* (though without naming him,) hath built the greatest part (if not the whole) of his *Algebra* or *Geometry*. Without which, that whole superstructure of *Des Cartes* (I doubt) had never been." Then on p. 198: "The Improvements of *Algebra* to be found in *Harriot* (as appears from what is already said,) and which (all or most of them) we owe to him; (of which it will not be amiss, before I leave him, to give a brief Recapitulation;) are chiefly these." Then follow 25 numbered statements of which no.

5 is: "Determining the Number of Roots (Affirmative, Negative, or Imaginary), in every equation. *viz.* So many as are the Dimensions of its Highest Term." This is contrary to HARRIOT's own statement in his *Artis*, pp. 89–90, Lemma, that equations have only positive roots. Directly and indirectly CAJORI shows that hardly any of WALLIS's statements quoted above are correct. But he also sums up the various contributions of importance which HARRIOT made. In H. W. TURNBULL, *Theory of Equations*, second ed., Edinburgh and London, 1944, reference is made (pp. 99–100) to "The Harriot-Descartes Rule of Signs" because the rule is "implicit in the work of Harriot". There is not the slightest foundation in fact for this statement.

180. For "Pascal on the arithmetic triangle" ed. by A. SAVITSKY, see SMITH [150], pp. 67–79; and for triangular arrays before the publication by PASCAL in 1654 see SMITH [4], v. 2, pp. 508–511.

181. F. CAJORI, (a) *William Oughtred, a Great Seventeenth-Century Teacher of Mathematics*, Chicago, 1916, 100 pp.; (b) "A list of Oughtred's mathematical symbols with historical notes," Univ. of California, *Publs. in Math.*, v. 1, pp. 171–186, 1920; (c) "On the history of Gunter's scale and the slide rule during the seventeenth century," *idem.*, pp. 187–209; (d) CAJORI [163], v. 1–2, see indices. AUBREY [162], Oughtred, v. 2, pp. 105–114. H. BOSMANS, "La première édition de la Clavis Mathematica d'Oughtred; son influence sur la géométrie de Descartes," Soc. Sci. de Bruxelles, *Annales*, v. 35, 1911, pp. 24–78; Quotation: "Celle-ci reste . . . un ouvrage original et de haute valeur, en progrès notable sur les travaux de Viète et ayant exercé une grande influence. Il n'est pas permis de la passer sous silence dans l'histoire des mathématiques."

181a. This publication is the interesting and remarkable anonymous Appendix (16 pp.) of the so-called second English edition by EDWARD WRIGHT, 1618, of Napier's *Descriptio*. The Appendix is reprinted in full in J. W. L. GLAISHER, "The earliest use of the radix method for calculating logarithms, with historical notices relating to the contributions of Oughtred and others to mathematical notation," *Quart. Jn. Math.* v. 46, pp. 125–197, 1915. The Oughtred abbreviations for the trigonometric functions are: s = sine, t = tangent, s_o = cosine, t_o = cotangent. In the Table are given the 54 values of $10^8 \log_e x$, for $x = [1(1)10(10)100(100)1000(1000)10\ 000(10\ 000)100\ 000\ (100\ 000)900\ 000; 6D]$; in the "Supplement of the Table for tenth and hundredth parts" are numbers from which the values for 18 more x's may be found. The errors of this table are listed in *Math. Tables and Other Aids to Computation*, v. 3, Jan. 1949.

182. F. CAJORI, *A History of the Logarithmic Slide Rule and Allied Instruments*. New York, 1909, viii, 128, x pp.

183. ARCHIBALD [57], pp. 15–16.

184. L. G. SIMONS, "Desargues on perspective triangles" and "Desargues on the 4-rayed pencil," SMITH [150], pp. 307–314. W. M. IVINS, JR., "A note on Girard Desargues," *Scripta Math.*, v. 9, 1943, pp. 33–48. G. VACCA, *Enciclopedia Italiana di Scienze Lettere ed Arti*, v. 12, 1931, p. 660. H. T. PLEDGE, *Science since 1500. A Short History of Mathematics, Physics, Chemistry, Biology*. London, 1939; "Kepler and Desargues," pp. 74–75.

185. BELL [53]; SMITH [54]; T. P. ARMSTRONG, "Pascal in England," *Notes and Queries*, London, v. 170, 8 Feb. 1936, pp. 102–103; A. WOLF, *A History of Science, Technology, and Philosophy in the 16 & 17th Centuries*. New York, 1935, pp. 223–225, 560–561 (picture of PASCAL's calculating machine). The definitive edition of the *Oeuvres* of Pascal is the one edited by L. BRUNSCHVICG, P. BOUTROUX, and F. GAZIER, 14 v., Paris, 1914–1925 (some v. second ed.). Biographies of PASCAL, his father and his sister are given pp. 1–164 of v. 1. H. BOSMANS, "Sur les Oeuvres mathématiques de Blaise Pascal," *Revue d. Questions Scientifiques*, 1929. É. PICARD, *Éloges et Discours Académiques*. Paris, 1931, "Pascal mathématicien et physicien," pp. 1–21. *Encyclopaedia of the Social Sciences*. New York, v. 12, 1934, "Pascal" by L. BRUNSCHVICG. A DESBOVES, *Étude sur Pascal et les Géomètres Contemporains, suivie de plusieurs Notes Scientifiques et Littéraires*. Paris, 1878, iv, 175 pp. +1 plate. W. H. BUSSEY, "The origin of mathematical induction," *Amer. Math. Monthly*, v. 24, 1917, pp. 199–297; "Pascal's use of complete induction," pp. 203–205. M. STUYVAERT, "Sur l'auteur de l'histoire de la roulette publiée par Blaise Pascal" *Bibl. Math.*, s. 3, v. 8, pp. 170–172, 1908.

186. ARCHIBALD [69], "limaçon."

187 PASCAL, *Oeuvres*, v. 1, pp. 243–260: "Essay pour les coniques," 1639, published on a single sheet 1640. Facsimile of this in *Isis*, v. 10, 1928 oppo. p. 16 and a translation by F. M.

CLARKE, pp. 17–20. In revised form this translation was given in SMITH [150], pp. 326–330. An earlier English edition was given by W. J. MACDONALD in Edinb. Math. Soc., *Proc.*, v. 2, 1884, pp. 19–24. And again by J. J. MILNE in *Math. Gazette*, v. 12, 1924, pp. 53–56.

188. D. BAXANDALL, *Catalogue of the Collection in the Science Museum. I. Calculating Machines and Instruments*. London, 1926, pp. 8, 12, 13. The Museum has a replica (made in 1925–26) of Pascal's original calculating machine. DIDEROT has given a detailed description of this machine and its use in *Encyclopédie Méthodique, Mathématiques*, v. 1, Paris, 1784, pp. 136–142. See also PASCAL [185], *Oeuvres*, v. 1, pp. 291–321+2 plates of the machine; SMITH [150], "Pascal on his calculating machine," ed. by L. L. LOCKE, pp. 165–172.

189. SMITH [150], "Fermat and Pascal on probability," ed. by V. SANFORD, pp. 546–565. I. TODHUNTER, *A History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace*. Cambridge and London, 1865, and New York reprint, 1931; "Pascal and Fermat," pp. 7–21, *etc.*

190. G. MOHR, *Euclides Danicus, Amsterdam, 1672*, with an introduction by J. HJELMSLEV, and a German translation from the Danish by J. PÁL. Copenhagen, 1928, 8, 36, 41 pp. +3 folding plates. This work contains a facsimile of the Danish original. There was also a Dutch edition in 1672. See also J. HJELMSLEV, ed., "Beiträge zur Lebensbeschreibung von Georg Mohr (1640–1697)," K. Danske Vidensk. Selskab, *mathem.-fysiske Meddelelser*, v. 11, no. 4, 1931, 22 pp.+1 plate.

191. R. E. LANGER, "René Descartes," *Amer. Math. Monthly*, v. 44, 1937, pp. 495–512. BELL [53], "Gentleman, soldier, mathematician—Descartes," pp. 35–55. I. B. HART, *Makers of Science. Mathematics, Physics, Astronomy*. London, 1923, "René Descartes and coördinate geometry," pp. 125–137. TURNBULL [51], pp. 70–76. SMITH [54], KEYSER [54]. G. S. MILHAUD, *Descartes, Savant*. Paris, 1921, 249 pp. E. S. HALDANE, *Descartes: His Life and Times*. New York, 1925, 398 pp. The standard edition of Descartes' *Oeuvres* is that edited by C. ADAM & P. TANNERY, 12 v., Paris, 1897–1910. In v. 12 there is a life of Descartes by ADAM, xix, 646 pp. Of the important *Descartes correspondance publiée avec une Introduction et des Notes* by C. ADAM & G. S. MILHAUD, 4 v. have so far appeared, Paris, 1937–1947, almost 400 pp. per v.

192. In the *Geometrie* steps in the demonstrations are often omitted and we meet with such statements as "I shall not stop to explain this in more detail because I should deprive you of the pleasure of mastering it yourself"; and "I find nothing here so difficult that it cannot be worked out by any one at all familiar with ordinary geometry and with algebra." This was done purposely by Descartes so that some professed mathematicians were unable to open their mouths in criticism of him, because they were unable to follow his arguments. These deficiencies were made good in the second edition of the *Geometrie*, in Latin, with notes and additions by F. DE BEAUNE and F. v. SCHOOTEN, published as a volume of 400 pp. at Amsterdam in 1659. There was an English edition, *The Geometry of René Descartes, Translated from the French and Latin* by D. E. SMITH and M. L. LATHAM, *with a Facsimile of the First Edition 1637*. Chicago, 1925, xiv, 246 pp. In the review of this v. by T. L. HEATH in *Nature*, v. 118, 1926, pp. 400–401 occurs the following: "The translation . . . is in many places inaccurate and sometimes wholly misleading."

This publication of DESCARTES gave him priority of publication over everyone else in connection with analytic geometry from the modern point of view. It is to be borne in mind, however, that several years before 1637 FERMAT had also independently come to the conception of an analytic geometry, and gone further than DESCARTES in consideration of tangents and maxima and minima of curves. Reference may be given to TROPFKE [11], v. 6, pp. 92–169; J. L. COOLIDGE (a) "The origin of analytic geometry," *Osiris*, v. 1, 1936, pp. 231–250; (b) *A History of Geometrical Methods*, Oxford, 1940, pp. 122–128. We know that DESCARTES read HARRIOT's work [179] but that his own work was thereby in any way benefited seems decidedly doubtful; but it is barely possible that he may have been influenced by OUGHTRED's *Clavis* [181]. J. TROPFKE, "Das x Symbol der unbekannten bei Descartes und seinen Nachfolgern," *Archeion*, v. 13, 1931, pp. 300–319.

193. ARCHIBALD [59], DESCARTES' Ovals and Folium.

194. BELL [53], "The prince of amateurs—Fermat," pp. 56–72. SMITH [54]. SMITH [150], "Fermat on analytic geometry," ed. by J. SEIDLIN, pp. 389–396; "Fermat and Pascal on probability,"

ed. by V. SANFORD, pp. 546–565; "Fermat on maxima and minima," ed. by V. SANFORD, pp. 610–612. TODHUNTER [189]. H. WIELEITNER, *Mathematische Quellenbücher*, v. 1: *Rechnen und Algebra*, Berlin, 1927, vi, 75 pp.; v. 2: *Geometrie u. Trigonometrie*, 1927, viii, 68 pp.; v. 3: *Analytische u. Synthetische Geometrie*, 1928, vii, 89 pp.; v. 4: *Infinitesimalrechnung*, 1929, vii, 160 pp.; "Fermat and the equations of a line and ellipse," v. 3, pp. 15–23; "Fermat's quadrature of higher hyperbolas and the method of maxima and minima," v. 4, pp. 82–96. G. MILHAUD, "Les querelles de Descartes et de Fermat au sujet des tangentes," *Rev. générale d. Sci.*, v. 28, 1917, pp. 332–337. J. F. SCOTT, *The Mathematical Work of John Wallis*. London, 1938: "Controversy with Fermat and other members of the French School," pp. 71–82. G. WERTHEIM, "Pierre Fermat's Streit mit John Wallis, ein Beitrag zur Geschichte der Zahlentheorie," *Abh. z. Gesch. d. Mathem.*, v. 9, 1899, pp. 555–576. CANTOR [12], v. 2, "Descartes, Fermat," pp. 851–876. The best edition of the *Oeuvres* of FERMAT is that edited by P. TANNERY and C. HENRY, 4 v., Paris, 1894–1912. ORE [7b]," The converse of Fermat's theorem," pp. 326–339.

195. T. L. HEATH, *Diophantus of Alexandria, a Study in the History of Greek Algebra*. Second ed., Cambridge, 1910; "Additional notes, theorems and problems of Fermat," pp. 267–328. J. M. CHILD, "Did Fermat have a solution of the so-called Pellian equation?", *Isis*, v. 3, 1920, pp. 255–262; CHILD concludes that FERMAT did, and that it was the one which he gives. DICKSON [101], many references in the index.

196. KLEIN [68], pp. 81–85.

197. A. DEMORGAN, "Cavalieri, Buonaventura," *Penny Cyclopaedia*, v. 6, 1836, and *English Cycl.—Biography*, v. 1, 1856. F. CAJORI, "Indivisibles and 'ghosts of departed quantities,'" *Scientia*, v. 37, 1925, pp. 301–306. *The Works of Aristotle translated into English under the editorship of* W. D. ROSS, v. 6, Oxford, 1913; "De lineis insectabilibus" (concerning indivisible lines), ed. by H. H. JOACHIM, Oxford, 1908, iv, 38 pp. H. BOSMANS, "Un chapitre de l'oeuvre de Cavalieri. Les propositions XV–XXVII de *l'Exercitatio quarta*," *Mathesis*, v. 36, 1922, pp. 365–373, 446–456. M. G. SITTIGNANI, "Sulla geometria degli indivisibile di B. Cavalieri," *Periodico d. Matematiche*, s. 4, v. 13, 1933, pp. 266–288. E. WALKER, *A Study of the Traité des Indivisibles of Gilles Persone de Roberval with a View to answering, insofar as possible, the two Questions: Which Propositions contained therein are his own, and which are due to his Predecessors or Contemporaries? and what Effect if any, had this work on his Successors?* New York, 1932, vi, 272 pp. ROBERVAL (1602–1675), not of noble birth but a native of the village of Roberval, for 49 years professor at the Collège de France, Paris, is best known by the *Traité* (1693), referred to above, where the subject is treated more scientifically than by CAVALIERI in a similar work (1635). See also H. M. WALKER, "An unpublished hydraulic experiment of Roberval, 1668," *Osiris*, v. 1, 1936, pp. 726–732. Information about EVANGELISTA TORRICELLI (1608–1647) may be found in *Encycl. Britannica*, eleventh ed., v. 27, 1911; and TORRICELLI, *Opere Matematiche*, 3 v., Fienza, 1919. C. R. WALLNER, "Die Wandlungen des Indivisibilienbegriffs von Cavalieri bis Wallis," *Bibl. Mathem.*, s. 3, v. 2, 1901, pp. 230–234. C. B. BOYER, *The Concept of the Calculus. A Critical and Historical Discussion of the Derivative and of the Integral*. New York, 1939, viii, 346 pp.

198. G. W. EVANS, "Cavalieri's theorem in his own words," *Amer. Math. Monthly*, v. 24, 1917, pp. 447–451. SMITH [150], "Cavalieri's approach to the calculus," ed. by E. WALKER, pp. 605–609.

199. *James Gregory. Tercentenary Memorial Volume. Containing his Correspondence with John Collins and His hitherto Unpublished Mathematical Manuscripts together with Addresses and Essays Communicated to the Royal Society* [of Edinburgh] *July 4, 1938*. Edited by H. W. TURNBULL, London, 1938, xii, 524 pp.+5 plates. The papers here include (pp. 468–509): M. DEHN & E. HELLINGER, "On James Gregory's *Vera Quadratura*"; E. J. DIJKSTERHUIS, "James Gregory and Christiaan Huygens" (controversy); A. PRAG, "On James Gregory's *Geometriae Pars Universalis*." A. M. CLERKE, *Dict. Nat. Biog.*, v. 23, 1890, pp. 98–99. F. CAJORI, "On an integration ante-dating the integral calculus," *Bibl. Math.*, s. 2, v. 14, 1915, pp. 316–318. G. HEINRICH, "James Gregory's *Vera Circuli et Hyperbolae Quadratura*," *Bibl. Math.*, s. 2, v. 2, 1901, pp. 77–85. H. W. TURNBULL, "James Gregory; a study in the early history of interpolation," Edinb. Math. Soc., *Proc.*, s. 2, v. 3, 1933, pp. 151–172. G. HEINRICH, "Notiz zur Geschichte der Simpsonschen Regel," *Bibl. Math.*, s. 2, v. 1,

1900, pp. 90–92; discussion of a passage in GREGORY's *Exercitationes Geometricae* (1668); parts of this work are published in F. MASERES, *Scriptores Logarithmici*, v. 2, 1796, pp. 1–19. G. GIBSON (a) "James Gregory's mathematical work: a study based chiefly on his letters," Edinb. Math. Soc. *Proc.*, v. 41, 1923, pp. 2–25; (b) "Sketch of the history of mathematics in Scotland to the end of the 18th century," *idem*, s. 2, v. 1, 1927, pp. 12–17.

200. Of the function u there have been many tables of which the first was by EDWARD WRIGHT (1550) in his *Certaine Errors in Navigation* with special reference to MERCATOR charts; see R. C. ARCHIBALD, Lambertian or lambda function," *Math. Tables and Other Aids to Computation*, v. 3, pp. 222–225, 1948.

201. Machin's result (in expanded form) was first published in W. JONES, *Synopsis palmiorum matheseos*, London, 1706, p. 263; it was also in this book, pp. 243, 263, that the symbol π for the ratio of the circumference of a circle to its diameter was first used. Making use of GREGORY's series for $\tan^{-1} x$, MACHIN found that if $\tan \alpha = \frac{1}{5}$, $\tan 4\alpha = \frac{120}{119}$, and that if $y = 4\alpha - \frac{1}{4}\pi$, $\tan y = \frac{1}{239}$; whence MACHIN's formula. The details of this derivation were first published by MASERES, *A Dissertation on the Use of the Negative Sign in Algebra*, London, 1758, pp. 289–290; see also MASERES, *Scriptores Logarithmici*, v. 3, 1796, pp. 157–161. It was WILLIAM SHANKS who computed π to 707*D*, and D. F. FERGUSON who discovered that this value was erroneous beyond 526*D*, and carried the computation along to 808*D*. Using MACHIN's formula JOHN W. WRENCH, JR., and LEVI S. SMITH checked the accuracy of this computation, details concerning which may be found in *Math. Tables and Other Aids to Computation* (*MTAC*), v. 2, 1947, pp. 245–248; v. 3, 1948, pp. 18–19. This calculation to 808*D* was intended as a companion to P. PEDERSEN's calculation of e to 808*D*; see *MTAC*, v. 2, pp. 68–69. There is a sketch of MACHIN, a professor at Gresham College, by A. M. CLERKE, in *Dict. Nat. Biog.*, v. 34, 1893.

202. A. DEMORGAN, "Huyghens, Christian," *Penny Cycl.*, v. 12, 1839, and *Engl. Cycl.—Biography*, v. 3, 1856. A. M. CLERKE, *Encycl. Brit.*, eleventh ed., v. 14, 1910. CREW [177]. P. LENARD, *Great Men of Science*, transl. from the second German ed., by H. S. HATFIELD. New York, 1933, "Huygens," pp. 67–83. A. E. BELL, *Christian* [*sic*] *Huygens and the Development of Science in the Seventeenth Century*. London, 1948, 220 pp.+portr., is a work of value. *The Contribution of Holland to the Sciences. A Symposium* edited by A. J. BARNOUW & B. LANDHEER, New York, 1943; "Astronomy" by JAN SCHILT, pp. 267–280; "Mathematics" by D. J. STRUIK, pp. 281–295. D. J. KORTEWEG, "La solution de Christiaan Huygens du problème de la chainette," *Bibl. Math.*, s. 3, v. 1, 1900, pp. 97–108. R. C. ARCHIBALD (a) "Discussion and history of certain geometrical problems of Heraclitus and Apollonius," Edinb. Math. Soc., *Proc.*, v. 28, 1910, pp. 152–178+5 plates; HUYGENS gave 13 solutions of these problems. (b) "Problems discussed by Huygens," *Amer. Math. Monthly*, v. 28, 1921, pp. 468–480. TODHUNTER [189], Huygens, pp. 22–25, etc. ARCHIBALD [69], catenary, cycloid, evolute. H. BOSMANS "Galilée ou Huygens. À Propos d'un épisode de la première application du pendule aux horlogues," *Rev. d. Questions Scient.*, s. 3, v. 22, 1912, pp. 573–586. The edition of Huygens' *Oeuvres*, of which 21 v. had been published by the Dutch Society of Sciences at Leiden, 1888–1944, is the most magnificent in appearance and detailed editing of the work of any scientist.

203. *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung. Deutsch herausgegeben und mit einer Übersicht über die Geschichte des Problems von den ältesten Zeiten bis auf unsere Tage, versehen von* R. RUDIO. Leipzig, 1892, viii, 166 p.; text statement on p. 40.

204. J. F. SCOTT, (a) *The Mathematical Work of John Wallis D.D., F.R.S. (1616–1703)*. London, 1938, xii, 240 p., portrait frontispiece. (b) "John Wallis as a historian of mathematics, *Annals of Science*, v. 1, 1936, pp. 325–337. A. DEMORGAN, "Wallis, John," *Penny Cycl.*, vol. 27, 1843, and *Engl. Cycl.—Biography*, v. 6, 1858. AUBREY [162], Wallis, v. 2, pp. 280–283. T. P. NUNN, "The Arithmetic of Infinities," *Math Gazette*, v. 5, pp. 345–346, 1910 and 377–386, 1911; an analysis of WALLIS's greatest work, *Arithmetica Infinitorum*, first published in 1655. We here find for the first time the familiar symbol, ∞, to denote infinity. SMITH [150], "Wallis on imaginary numbers" ed. by D. E. SMITH, pp. 46–54; "Wallis on general exponents" ed. by E. M. SANFORD, pp. 217–218; "Wallis and Newton on the binomial theorem for fractional and negative exponents," ed. by D. E. SMITH, pp. 219–223. G. ENESTRÖM, "Die geometrische Darstellung imaginärer Grössen bei Wallis,"

Bibl. Math., s. 3, v. 7, 1907, pp. 263–269. A. CAYLEY, "The investigation by Wallis of his expression for π," *Quart. Jn. Math.*, v. 23, 1889, pp. 165–169, and *Math. Papers*, v. 13, 1897, pp. 22–25. G. A. DICKINSON, "Wallis's product for $\frac{1}{2}\pi$," *Math. Gazette*, v. 21, 1937, pp. 135–139. H. WIELEITNER, "Die Verdienste von John Wallis um die analytische Geometrie" *Das Weltall*, v. 29, 1930, pp. 56–60. W. LOREY, "Bruchaufgaben und Reihensumme nach Wallis," *Unterrichtsblätter f. Mathem. u. Naturw.*, v. 41, 1935, pp. 57–58. W. KUTTA, "Elliptische und andere Integrale bei Wallis," *Bibl. Math.*, s. 3, v. 2, 1901, pp. 230–234. A. PRAG, "John Wallis 1616–1703. Zur Ideengeschichte der Mathematik im 17. Jahrhundert," *Quellen und Studien* . . . , v. 1B, 1930, pp. 381–412. F. CAJORI, "Controversies on mathematics between Wallis, Hobbes, and Barrow," *Math. Teacher*, v. 22, 1929, pp. 146–151. D. E. SMITH, "John Wallis as a cryptographer," Amer. Math. Soc. *Bull.*, v. 24, 1917, pp. 82–96; WALLIS seems to have served his country as expert cryptographer for 60 years; VIETA—gave similar service in France. WALLIS'S *Opera Mathematica*, 3 large v., Oxford, 1695, 1693, 1699.

Apart from the circle, the first three curves (see ARCHIBALD [69]) to be rectified were as follows: logarithmic spiral by TORRICELLI in 1640; semi-cubical parabola by WILLIAM NEILE (a pupil of Wallis and a friend of CHRISTOPHER WREN) in 1657; and cycloid by WREN in 1658. Neile's rectification was based on principles set forth in Wallis's *Arithmetica Infinitorum*. In 1657 HUYGENS first showed that the rectification of a parabola depended upon the quadrature of a rectangular hyperbola. Learning of this result, HENDRIK VAN HEURAET, a Dutch disciple of DESCARTES, made advances, published in the 1659 edition of DESCARTES' geometry. The names of NEWTON, FERMAT, GREGORY also come up in the admirable survey of J. E. HOFMANN, "Über die ersten logarithmischen Rektifikationen. Eine historisch-kritische Studie in vergleichender Darstellung," *Deutsche Math.*, v. 6, 1941, pp. 283–303. This article is based on his earlier important discussions: "Nicolaus Mercators Logarithmotechnia (1669)" and "Weiterbildung der logarithmischen Reihe Mercators in England," *Deutsche Math.*, v. 3, 1938, pp. 446–466, 598–605; v. 4, 1939, pp. 556–562; v. 5, 1940, pp. 358–375. JAMES GREGORY in 1667 seems to have been the first to show (or at least to publish the result) that if $xy=1$ is the equation of a rectangular hyperbola, the area between the two ordinates $y=1$ and $y=a$, is $\log_e a$; see G. ENESTRÖM, *Bibl. Math.*, s. 3, v. 11, 1911, p. 239.

205. HISTORY OF SCIENCE SOCIETY, ed., *Sir Isaac Newton, 1727–1927. A Bicentenary Evaluation of His Work*, Baltimore, 1928, 9+351 pp. [Contents: "Newton in the light of modern criticism" by D. E. SMITH, pp. 3–11; "Newton and optics" by D. C. MILLER, pp. 15–48; "Newton's philosophy of gravitation with special reference to modern relativity ideas" by G. D. BIRKHOFF, pp. 51–64; "Newton's influence upon the development of astrophysics" by W. W. CAMPBELL, pp. 67–86; "Newton's dynamics" by M. I. PUPIN, pp. 89–97; "Newton as an experimental philosopher" by P. R. HEYL, pp. 101–108; "Developments following from Newton's works" by E. W. BROWN, pp. 111–124; "Newton's twenty years' delay in announcing the law of gravitation" by F. CAJORI, pp. 127–188; "Newton's fluxions" by F. CAJORI, pp. 191–200; "Newton's work in alchemy and chemistry" by L. C. NEWELL, pp. 203–255; "Newton's place in the history of religious thought" by G. S. BRETT, pp. 259–273; "Newton in the Mint" by G. E. ROBERTS, pp. 277–298; "Newton's first critical disciple in the American colonies—John Winthrop" by F. E. BRASCH, pp. 301–338; . . . Material displayed in the Newton exhibit at the Amer. Museum of Nat. Hist., pp. 342–351.]

206. MATHEMATICAL ASSOCIATION, England, ed., *Isaac Newton, 1642–1727. A Memorial Volume*, London, 1927, 8+181 pp.+plates. [Contents: "Two unpublished documents of Newton" by D. E. SMITH, pp. 16–34; "Letters from Newton in Corpus Christi College, Oxford," ed. by J. L. E. DREYER, pp. 35–44; "Newton and interpolation" by D. C. FRASER, pp. 45–69; "Newton's work in optics" by E. T. WHITTAKER, pp. 70–74; "Newton's problem of the solid of least resistance" by A. R. FORSYTH, pp. 75–86; "Newton's work on the theory of the tides" by J. PROUDMAN, pp. 87–95; "Newton's contribution to the geometry of conics" by J. J. MILNE, pp. 96–114; "Newton on plane cubic curves" by H. HILTON, pp. 115–116; "Newton and the art of discovery" by J. M. CHILD, pp. 117–129; "Newton's influence on method in the physical sciences" by A. E. HEATH, pp. 130–133; "Plagiarism in the seventeenth century, and Leibniz" by L. J. RUSSELL, pp. 134–136; "The contemporary significance of Newton's metaphysics" by E. A. BURTT, pp. 137–140; "Newton and his homeland; the haunts of his youth" by J. A. HOLDEN, pp. 141–143; "Trinity College in the time of Newton" by G. N. WATSON, pp. 144–147; "A Newton Bibliography" by H.

ZEITLINGER, pp. 148–170; "Portraits of Sir Isaac Newton" by D. E. SMITH, pp. 171–178; "The portrait medals of Newton" by D. E. SMITH, pp. 179–180.]

207. THE ROYAL SOCIETY *Newton Tercentenary Celebrations 15–19 July 1946.* Cambridge, University Press, 1947, xvi, 92 p.+6 plates (3 portraits of Newton, Newton's rooms at Trinity, Woolsthorpe Manor, Newton letter). [Contents: "Address of welcome to the delegates" by R. ROBINSON, President of the R. S., pp. 1–2; "Newton" by E. N. DA C. ANDRADE, pp. 3–23; "Address of welcome to the delegates" by G. M. TREVELYAN, Master of Trinity, pp. 24–26; "Newton, the man" by the late *Lord* KEYNES (in 1942), pp. 27–34; "Newton and the infinitesimal calculus" by J. HADAMARD, pp. 35–42; "Newton and the atomic theory" by S. I. VAVILOV, pp. 43–55; "Newton's principles and modern atomic mechanics" by N. BOHR, pp. 56–61; "Newton: the algebraist and geometer" by H. W. TURNBULL, pp. 62–72; "Newton's contributions to observational astronomy" by W. ADAMS, pp. 73–81; "Newton and fluid mechanics" by J. C. HUNSAKER, pp. 82–90.]

208. R. E. LANGER, "Isaac Newton," *Scripta Mathem.*, v. 4, pp. 241–255, 1936. H. W. TURNBULL, *The Mathematical Discoveries of Newton.* London and Glasgow, 1945, viii, 68 pp.+frontispiece portr.; admirable little book. W. W. BALL, "Newton," *Math. Gazette*, v. 7, pp. 349–360, 1914. CREW [177]. HART [60], pp. 138–172. BELL [53], "On the seashore—Newton," pp. 90–116. SMITH [54]. A. DEMORGAN, *Essays on the Life and Work of Newton. Edited with Notes and Appendices* by P. E. B. JOURDAIN. Second ed., Chicago, 1914, xiv, 198 pp. S. BRODETSKY, *Sir Isaac Newton a Brief Account of His Life and Work.* London, 1927, xii, 161 pp.+plate. J. CRAIG, *Newton at the Mint.* Cambridge, Engl., 1946, vi, 128 pp., 4 plates, J. W. N. SULLIVAN, *Isaac Newton, 1642–1727. . . . With a memoir of the author* by C. SINGER. London, 1938, xx, 275 pp.; very attractively written. L. T. MORE, *Isaac Newton, a Biography.* New York and London, 1934, xiv, 675 pp.; of the long biographies of Newton this is a decided advance on Brewster's, but since the author is hampered by the fact that he is out of sympathy with modern developments in physics, he fails to see Newton's work in perspective.

In R. DE VILLAMIL, *Newton: The Man.* London, 1931, vi, 112 p.+portrait frontispiece, is a list of 1896 books originally in Newton's library. (Brown University has the fine copy of DEMOIVRE's *Annuities upon Lives*, 1725, here listed.) In July 1943, 860 of these volumes, including the septuagint Old Testament in Greek, and BARROWS' edition of EUCLID's Elements, were purchased by The Pilgrim Trust and presented to the Library of Trinity College, Cambridge, which had been founded by ISAAC BARROW, who, before resigning his chair (1669) in NEWTON's favor, and later becoming Master of Trinity, declared that he could not make a Bible out of his EUCLID or a pulpit out of his mathematical chair. See *The Times*, London, 1943, Apr. 12, p. 6d; 14, p. 5d; June 30, p. 2c. In "Sir Isaac Newton's early study of the Apocalypse," *Pop. Astron.*, v. 34, 1926, pp. 75–78, F. CAJORI describes NEWTON's much annotated copy of HENRY MORE's work in the Library of the University of California at Berkeley.

Newton's last London residence (1710–1725) on St. Martin's St., Leicester Square, was demolished in 1913 by order of the London City Council, but the woodwork of the "fore parlour" was in 1939 re-erected in America, in the Library of the Babson Institute, BABSON Park, Mass., where there is also a large collection of Newtoniana. A picture of this "parlour" appears in F. E. BRASCH, "Newton's portraits and statues," *Scripta Math.*, v. 8, 1941, pp. 199–227, 24 of the pages being 12 plates, 10 of which are portraits. See also "Sale of Newtoniana," *Nature*, v. 138, 1936, p. 195.

An interesting item is MARJORIE H. NICOLSON, *Newton Demands the Muse; Newton's Opticks and the Eighteenth Century Poets.* Princeton, N. J., 1946, xi, 177 pp.

209. F. CAJORI, *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse.* Chicago and London, 1919, viii, 299 pp.+frontispiece. A. DEMORGAN, "Fluxions, fluents, method, notation and early history," *Penny. Cycl.*, v. 10, 1838. F. CAJORI, "The spread of Newtonian and Leibnizian notations of the calculus," Amer. Math. Soc., *Bull.*, v. 27, 1921, pp. 453–458. G. A. GIBSON, (a) "Berkeley's Analyst and its critics: an episode in the development of the doctrine of limits," *Bibl. Math.*, s. 2, v. 13, 1899, pp. 65–70; (b) Edinb. Math. Soc., *Proc.*, v. 17, 1899, pp. 9–32. "Berkeley and Newton," *Math. Gazette*, v. 7, pp. 418–421, 1914. GEORGE BERKELEY (1685–1753), bishop of Cloyne, mathematician, and one of the most subtle and original meta-

physicians, has considerable interest for both American and mathematical history. He lived in Newport, Rhode Island, Jan. 1729–Aug. 1731 and there wrote his longest work *Alcephron or the Minute Philosopher* (publ. 1732). He was raised to the bishopric of Cloyne in 1734. His first printed work (1707) consisted of two mathematical tracts. In 1734 he published *The Analyst: or a Discourse addressed to an infidel Mathematician* [Halley]. *Wherein it is examined whether the Object, Principles, and Inferences of the Modern Analysis are more distinctly conceived, or more evidently deduced, than religious Mysteries and Points of Faith. By the Author of The Minute Philosopher.* This precipitated the *Analyst* controvery in which many participated. See "Bibliography of George Berkeley," *Scripta Math.*, v. 3, 1935, pp. 81–83.

SMITH [150], "Wallis and Newton on the binomial theorem for fractional and negative exponents," ed. by D. E. SMITH, pp. 219–223; "Newton on the binomial theorem for fractional and negative exponents," ed. by E. M. SANFORD, pp. 224–228; "Newton on fluxions," ed. by E. WALKER, pp. 613–618.

210. A. DEMORGAN, "Barrow, Isaac," *Penny Cycl.*, v. 3, 1835, and *English Cycl.—Biography*, v. 1, 1856. The Soc. for the Promotion of Christian Knowledge published a popular biography, with slighting of scientific matters: P. H. OSMOND, *Isaac Barrow, His Life and Times*. London, 1944, vi, 230 pp., 4 plates. AUBREY [162], BARROW, v. 1, pp. 87–94. BARROW was professor of geometry at Gresham College (1662–64) and first Lucasian professor of mathematics at Cambridge (1664–69). His *Mathematical Works* (Cambridge, 1860, xx, 414, ii, 320 pp., 27 folding plates) were edited by W. WHEWELL, and his *Geometrical Lectures . . . Translated, with Notes and Proofs, and a Discussion on the Advance made therein on the Work of his Predecessors in the Infinitesimal Calculus*, edited by J. M. CHILD, was published at Chicago, 1916, xiv, 218 pp.

211. The composition and publication of the *Principia* was due to a young man about thirty years of age, **EDMOND HALLEY**, who became the successor of WALLIS as Savilian professor of geometry at Oxford. He has been called "flapper HALLEY" that is, according to SWIFT, one who flaps another to remind him of a task to be performed. The huge task of seeing the work through the press involved him in considerable personal financial outlay. The first edition was very small and probably sold for ten or twelve shillings a copy. It went out of print so quickly it was difficult to obtain even in 1691. Before World War II a copy could be occasionally picked up for $100, but recently five times that amount has been obtained. Many corrections and developments of the first edition occurred in the second edition published in 1713 (750 copies), which sold unbound at fifteen shillings a copy. (Of this edition there were pirated Amsterdam reprints in 1714 and 1723.) There was a reprint of the second edition at Glasgow in 1871. For interesting information about the second ed., see W. G. HISCOCK (a) "The war of the scientists; new light on Newton and Gregory [David]," *Times Lit. Suppl.*, v. 35, 1936, p. 34; (b) *David Gregory, Isaac Newton and their Circle.* Oxford, 1937, ix, 48 pp. The printing of the third edition begun in 1723, when NEWTON was 80 years old, was not finished till 1726, the year before he died. The student of the *Principia* may find the following useful: W. W. R. BALL, *An Essay on Newton's 'Principia.'* London, 1893, x, 175 pp. The *Principia* was first translated into English (from the third ed.) by ANDREW MOTTE (1729); R. THORPE published a translation of book I (1777). These are the bases of the English ed. by F. CAJORI, posthumously published by the Univ. of California in 1934, xxvi, 680 pp.; second printing 1946. A rather scathing review by R. A. SAMPSON appeared in *Nature*. v. 135, 1935, pp. 128–129. See also *Scripta Math.*, v. 3, 1935, pp. 69–74, 186–187; and *Eureka, The Archimedean's Jn.*, no. 10, Mar. 1948, pp. 3–5. To the second edition of the *Principia*, HALLEY contributed a Latin poem entitled (in translation): "To this mathematico-physical work of the illustrious Mr. Isaac Newton, an achievement which is the greatest glory of our age and nation." Of this poem four English editions have been published: (1). 1759, by "Eugenio," in *General Mag. of Arts and Sciences*, London, v. 1, p. 4, and E. F. MACPIKE, *Correspondence and Papers of Edmond Halley*. Oxford, 1932, pp. 207–208. (2). 1934, by L. J. RICHARDSON, in Cajori's edition of the *Principia*, noted above, pp. xiii–xv. (3). Jan. 1943, by A. Weinstein, *Science*, n.s., v. 97, p. 70. (4). Aug. 1943, by Mrs. S. CHAPMAN, *Nature*, v. 152, p. 231. See also R. C. ARCHIBALD, "Mathematicians, and poetry and drama," *Science*, n.s., v. 89, 1939, pp. 48–49, footnote 74.

Good short sketches of Halley are those of A. DEMORGAN, *Penny Cycl.*, v. 12, 1839; or *English*

Cycl.—Biog., v. 3, 1856; and A. M. CLERKE, *Dict. Nat. Biog.*, v. 24, 1890, pp. 104–109. But the most comprehensive and up-to-date source is *Correspondence and Papers of Edmond Halley, preceded by an Unpublished Memoir of his Life by one of his Contemporaries and the Éloge by D'Ortous de Mairan, arranged and edited* by E. F. MACPIKE. Published for the [Amer.] History of Science Society, Oxford, Clarendon Press, 1932, xiv, 300 pp., 11 plates. We have already referred [86, 103] to publications of HALLEY editions of works by APOLLONIUS and MENELAUS. He was also a notable contributor to the foundation of statistical and actuarial theory (see H. BÖCKH, "Halley als Statistiker," Inst. Intern. de Statistique, *Bull.*, Rome, v. 7, 1893, pp. 1–24; and C. WALFORD, *Insurance Cyclopaedia*, London, v. 5, 1878, pp. 616–618, and v. 1, 1871, pp. 360–362). One of his outstanding contributions in the field of astronomy was when, in 1682, he identified, and computed the orbit of a comet with a period of about 76 years, last seen in 1910. He observed that the orbit was nearly identical with those of the comets in 1607 and 1531, and he also found that this was the same comet which was recorded as appearing in 1456, 1301, 1145, 1066. Compare *Pop. Astron.*, v. 42, 1934, pp. 191–201, where 29 "observations," 240 B.C.–1910 A.D. are discussed.

212. "Gauss's line" is a term also used in this connection since GAUSS showed (Zach's *Monatliche Korrespondenz*, 1810, p. 120; GAUSS, *Werke*, v. 4, pp. 385–392; see also *Mess. Math.*, v. 4, 1868, p. 137) that the locus of the centers of ellipses tangent to the sides of a complete quadrilateral is the line through the centers of the three diagonals. TERQUEM noted in 1845 (*Nouv. Annales de Math.*, v. 4, p. 545) that the five NEWTON lines, of the quadrilaterals formed by omitting one of five tangents to a conic, are concurrent. That the middle points of the three diagonals of a complete quadrilateral are collinear was first pointed out by F. J. CONNOR in *Ladies Diary*, 1795, p. 47.

213. *Sir Isaac Newton's Enumeration of Lines of the Third Order, Generation of Curves by shadows, . . . , translated from the Latin with notes and examples*, by C. R. M. TALBOT, London, 1861, xii, 5–140 pp., 12 folding plates; W. W. R. BALL, "Newton's classification of cubic curves," London Math. Soc., *Proc.*, v. 22, pp. 104–143, 1890; *Encykl. d. math. Wissen.*, v. III_2, pp. 461–462. See also R. H. GRAHAM, "Newton's influence on modern geometry," *Nature*, v. 42, 1890, pp. 139–142; C. TAYLOR, "On a section of Newton's Principia in relation to modern geometry," Cambridge Phil. Soc., *Proc.*, v. 3, 1880, pp. 359–360.

214. The memoranda of DAVID GREGORY for the period 1696–1708 give interesting information about the publication of the first edition of *Arithmetica Universalis*, edited and published by WM. WHISTON in 1707; see HISCOCK [211]. Other Latin editions were published in 1732 and 1761, at London, Leiden, and Milan. The first English edition appeared in 1720, the second in 1728, and another in 1769. There were also a French edition (1802) and various volumes of illustrations. For further details see G. J. GRAY, *A Bibliography of the Works of Sir Isaac Newton*, second ed. rev. and enl., Cambridge, Engl., 1907.

215. The most remarkable passage in Newton's *Arithmetica Universalis* is the Rule for the discovery of Imaginary Roots, stated without any proof (see H. W. TURNBULL, *The Mathematical Discoveries of Newton*, 1945, pp. 49–51). How NEWTON discovered such a complicated Rule is a mystery. It was not until nearly 160 years later that a proof of the rule was finally discovered by J. J. SYLVESTER, in 1864–65; see R. Soc. London, *Trans.*, v. 154, 1864, pp. 579–666+2 pls. and *Proc.*, v. 13 (1863–64), pp. 179–183, and v. 14 (1865), pp. 268–270; also in SYLVESTER, *Coll. Math. Papers*, v. 2, Cambridge, 1908, pp. 376–479, 493–494. See also the summary by H. J. PURKISS, *Oxford, Cambr. and Dublin Mess. Math.*, v. 3, 1865, pp. 129–142. The Rule is a generalization of DESCARTES' Rule of Signs, but it is a much deeper result. In a leaflet of 1896 SYLVESTER noted

"Descartes' and Newton's law lay hid in night:
Heaven touched my heart with fire, and all was light."

The American poet SIDNEY LANIER, in his "Ode to The Johns Hopkins University" has the following (lines 10–14):

"From far the sages saw, from far they came
And ministered to her,
Led by the soaring-genius'd Sylvester
That, earlier, loosed the knot great Newton tied,
And flung the door of Fame's locked temple wide."

F. CAJORI has given "A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity," Colorado College, *Publication, General Series*, nos. 51–52, *Science Series*, v. 12, 1910, pp. 171–287. The first part, before VIETA, pp. 171–182; the second part, pp. 182–215, deals with VIETA, NEWTON, LAGRANGE; the third part, pp. 217–288, modern times, discusses the NEWTON-RAPHSON method and allied processes, RUFFINI-HORNER method, WEDDLE'S method, DANDELIN-GRÄFFE method, method by infinite series, etc. See also the rather trivial M. A. NORDGAARD, *A Historical Survey of Algebraic Methods of Approximating the Roots of Numerical Higher Equations up to the Year 1819*. Diss. Columbia, Teachers College, New York, 1922, vi, 65 pp. "Vieta's extension of the Hindu method of approximating to complete equations," pp. 24–32; "Newton's method of approximation," pp. 33–46; "Certain effective but non-practicable methods," pp. 47–52; "Horner's method, similar methods by Ruffini and by the Chinese of the thirteenth century," pp. 53–57. F. CAJORI, "Newton's solution of numerical equations by means of slide rules," Colorado College, *Publication, General Series* no. 95, *Engineering Series*, v. 1, no. 18, 1917, pp. 245–253. C. RUNGE, "Separation und Approximation der Wurzeln," *Encykl. d. mathem. Wissen.*, v. 1, pp. 404–448, 1899. The so-called "Gräffe method" is really due to earlier writers: EDWARD WARING (1734–1798), *Meditationes Analyticae*, 1776, pp. 311 f., or GERMINAL DANDELIN (1794–1847), "Recherches sur la résolution des équations numériques," Acad. roy. d. Sci. d. Belgique, *Mémoires*, v. 3, 1826—pp. 7–71, 153–159, but especially p. 48 f. The method was also suggested independently by N. I. LOBACHEVSKY (1793–1856), *Algebra ili Ischislenie Konechnykh Velichin* [Algebra or Calculus of Finite Quantities]. Kazan, 1834, 528 p.; p. 157, §257. A few other accounts mainly expository, but some expansive, are: E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, third ed., seventh impression, London, 1940, pp. 106–120. S. BRODETSKY and G. SMEAL, "Method for complex roots of algebraic equations," Camb. Phil. Soc., *Proc.*, v. 22, pp. 83–87, 1924. C. A. HUTCHINSON, *Amer. Math. Monthly*, v. 42, 1935, pp. 149–163. D. H. LEHMER, "The Graeffe process as applied to power series," *Math. Tables and Other Aids to Comput.*, v. 1, pp. 377–383, 1945, and v. 3, pp. 227, 1948.

It was the Belgian professor of Physics, G. P. DANDELIN, referred to above, who discovered the beautiful theorem that the foci of the conic cut by a plane, p, from a right circular cone, are the points of contact of the spheres tangent to the cone and to p. Furthermore, the planes of the contact circles of the spheres intersect p in the directrices of the conic. See DANDELIN, Acad. d. Sci. d. Bruxelles, *Nouv. Mém.*, v. 2, 1822, pp. 171–173.

216. As yet I have not been able to trace this quotation to its source: the earliest reference I can give is to F. R. MOULTON, *Introduction to Astronomy*, New York, 1906, p. 199. If this story is true it would seem as if the Queen in question must have been SOPHIA DOROTHEA, wife of FREDERICK WILLIAM I, and mother of FREDERICK the Great. She was the first Queen of Prussia during the last 10 years that LEIBNIZ lived.

217. BELL [53], "Master of all trades—Leibniz," pp. 117–130. SMITH [54]. KEYSER [54]. CAJORI [4], pp. 205–219, etc. LENARD [202], "Leibniz," pp. 111–118. SMITH [150], "Leibniz and the Bernoullis on the polynomial theorem," ed. by J. GINSBURG, pp. 229–231; "Leibniz on the calculus," ed. by E. WALKER, pp. 619–626. J. M. CHILD, *The Early Mathematical Manuscripts of Leibniz. Translated from the Latin Texts published by Carl Immanuel Gerhardt with critical and historical Notes*. Chicago and London, 1920, iv, 238 pp. F. CAJORI. "Grafting the theory of limits on the calculus of Leibniz," *Amer. Math. Monthly*, v. 30, 1923, pp. 223–234. BOYER [197], "Newton and Leibniz," pp. 187–223.

218. E. STEPHENS, "Bibliography on general (or fractional) differentiation," Washington Univ., *Studies, Scientific Series*, v. 12, pp. 149–152, 1925.

219. T. MUIR, (a) *The Theory of Determinants in the Historical Order of its Development*, v. 1, up to 1841, second ed. London, 1906, xii, 491 pp.; v. 2, 1841–1860, 1911, xvi, 476 pp.; v. 3, 1861–1880, 1920, xxvi, 503 pp.; v. 4, 1880–1900, 1923, xxxi, 508 pp. (b) *Contributions to the History of Determinants*, [v. 5], 1906–1920, 1930, xxiv, 408 pp. Leibniz, v. 1, pp. 6–10. SMITH [150], "Leibniz on determinants," ed. by T. F. COPE, pp. 267–270.

220. L. L. LOCKE, "The contribution of Leibniz to the art of mechanical calculation," *Scripta Math.*, v. 1, 1933, pp. 315–321+1 plate. SMITH [150], "Leibniz on his calculating machine," ed. by M. KORMES, pp. 173–181. LEIBNIZ began work on the calculating machine about 1671 and had

completed one machine in 1694 and a second about 1706. See R. MEHMKE & M. D'OCAGNE, *Encycl. d. Sci. Math.*, tome 1, v. 4, fasc. 2, 1908, pp. 248–251.

221. F. CAJORI, "Leibniz the master-builder of mathematical notations," *Isis*, v. 7, 1925, pp. 419–429. D. MALMKE, "Leibniz als Begründer der symbolischen Mathematik," *Isis*, v. 9, 1927, pp. 279–293. Also CAJORI [163], see indices, v. 1–2.

222. A. DEMORGAN, *Penny Cycl.*, v. 4, 1835, "Bernoullis." BELL [53], "Nature or Nurture?—The Bernoullis," pp. 131–138. *Encycl. Britannica*, eleventh ed., v. 3, 1910, pp. 803–805. James=Jacques=Jakob=Giacomo; John=Jean=Johann=Giovanni. *Gedenkbuch der Familie Bernoulli zum 300. Jahrestage ihrer Aufnahme in das Bürgerrecht 1622–1922*. Basel, 1922, viii, 287 pp.+4 plates. Interesting portraits, facsimiles of writing, etc. C. v. BEHR-PINNOW, "Begabungsvererbung der Familie Bernoulli," *Naturwissenschaften*, v. 22, 1934, pp. 717–721. R. WOLF, *Biographien zur Kulturgeschichte der Schweiz*, 4 v. Zürich, v. 1, 1858, pp. 133–166. CANTOR [12], v. 3, "Jakob und Johann Bernoulli," pp. 89–98, 215–248, etc. JAMES BERNOULLI, *Unendliche Reihen (1689–1704) . . . aus dem Lateinischen übersetzt und herausgegeben*, by G. KOWALEWSKI. (*Ostwald's Klassiker* series). Leipzig, 1909, 142 pp. A. PROCISSI, "Il Problema bernoulliano 'De quadrisectione trianguli scaleni per duas normales rectas,'" *Per. d. Matem.*, s. 4, v. 14, 1934, pp. 1–27. This problem was propounded and solved by BERNOULLI in *Acta Eruditorum*, 1687, pp. 617–623. It was also solved by L'Hospital before 1704. See LORIA's history of the problem, *Bibl. Math.*, s. 3, v. 4, 1903, pp. 48–51.

223. See, for example, D. N. LEHMER, "Cornu's Spiral as transition curve," *California Jn. Technology*, v. 3, 1904, pp. 71–82. A. L. HIGGINS, *The Transition Spiral and its Introduction to Railway Curves*. London, 1921, viii, 112 pp. A. N. TALBOT, *The Railway Transition Spiral*. Sixth ed. New York, 1927, viii, 96 pp.

224. R. C. ARCHIBALD, "Euler's integrals and Euler's spiral—sometimes called Fresnel integrals and the clothoide or Cornu's spiral," *Amer. Math. Monthly*, v. 25, 1918, pp. 276–282.

225. R. C. ARCHIBALD, "The logarithmic spiral," pp. 146–151 of J. HAMBIDGE, *Dynamic Symmetry*. New Haven, 1920.

226. TODHUNTER [189], "James Bernoulli," pp. 56–77; "Daniel Bernoulli," pp. 213–238, etc. Of BERNOULLI'S *Ars Conjectandi* there was a French edition of the first part, "*L'art de Conjecturer*." Caen, 1801, iv, 180 p.; "Observations, eclaircissements et additions" occupy pp. 101–180. The German edition of all four parts, *Wahrscheinlichkeitsrechnung* (*Ars Conjectandi*), transl. and ed. by R. HAUSSNER, appeared in the *Ostwald's Klassiker* series, 2 v., Leipzig, 1899, 162, 172 pp. An English translation of the first three chapters of the second book of the *Ars*, by F. MASERES appeared in his *The Doctrine of Permutations and Combinations*. London, 1795, pp. 35–213; also in MASERES, *Scriptores Logarithmici*. London, v. 3, 1796, pp. 25–133 (pp. 100–133 being "An appendix to the foregoing translation").

227. J. L. COOLIDGE, "The beginnings of analytic geometry of three dimensions," *Amer. Math. Monthly*, v. 55, 1948, pp. 76–86.

228. *Euclides ab omni naevo vindicatus*, Milan, 1733, 142 pp. SACCHERI's work was translated into English (book 1; book 2, pp. 102–142 omitted) and edited by G. B. HALSTED, Chicago, 1920, xx, 246 pp. E. MCCLINTOCK, "On the early history of the non-euclidean geometry," N. Y. Math. Soc., *Bull.*, v. 2, pp. 144–147, 1893. R. BONOLA, *Non-Euclidean Geometry. A Critical and Historical Study of its Development*. Engl. transl. by H. S. CARSLAW. Chicago, 1912, pp. 22–44, etc. E. CARRUCCIO, "Saccheri, Giovanni Girolamo," *Enciclopedia Italiana*, Rome, v. 30, 1936.

229. HELEN M. WALKER, "Abraham De Moivre," *Scripta Math.*, v. 2, pp. 311–333, 1934; exceedingly interesting and comprehensive sketch. A. M. CLERKE, "Abraham De Moivre," *Dict. Nat. Biog.*, v. 38, 1894, pp. 116–117. C. HUTTON, *Phil. and Math. Dict.*, second ed., London, 1815, v. 2, p. 402. C. WALFORD, *The Insurance Cyclopaedia*, London, v. 1, 1871, "Annuities on Lives, Hist. of," pp. 120–122; v. 2, 1873, "De Moivre" and "De Moivre's hypothesis," pp. 180–183. A. DEMORGAN, "Demoivre, Abraham" and "Demoivre's hypothesis," *Penny Cycl.*, v. 8, 1839, pp. 380–381. D. E. SMITH, "Among my autographs: De Moivre expresses himself," *Amer. Math. Monthly*, v. 29, 1922, pp. 340–343. H. M. WALKER, *Studies in the History of Statistical Method, with Special Reference to Certain Educational Problems*, Baltimore, 1929, viii, 229 pp.; "De Moivre," pp. 12–19, *etc.*

230. TODHUNTER [189], pp. 134–193, etc. SMITH [150], "De Moivre on the laws of normal probability," ed. by H. M. WALKER, pp. 566–575. See also [229].

231. The formula in question is the following: If m is very large

$$m! \approx (2\pi m)^{1/2} e^{-m} m^m. \tag{1}$$

This is due to DE MOIVRE (1730–1733) and was not anywhere given by STIRLING or implied by anything he stated. We shall briefly indicate STIRLING's small contribution in the direction of this result. JAMES STIRLING (1692–1770) was a prominent Scottish mathematician who wrote two books: A commentary on NEWTON's lines of the third order, *Lineae tertii ordinis Newtonianae sive illustratio tractatus Newtoni de enumeratione linearum tertii ordinis.* Oxford, 1717, and added two new kinds (J. P. DE GUA subsequently detected four others) to the 72 which NEWTON had remarked among curves of the third order. At p. 32 of this work occurs the theorem which usually goes by the name of MACLAURIN's Theorem. His second, and most important book was one on the calculus of finite differences. *Methodus differentialis: sive Tractatus de Summatione et Interpolatione Serierum Infinitarum.* London, 1730, viii, 154 pp. There was a second edition in 1764 and an English edition in 1749. Information about his life and work may be found in A. DEMORGAN, *Engl. Cycl.-Biog.*, v. 5, 1857, "Stirling"; and C. TWEEDIE, (a) "Life of James Stirling, the Venetian," *Math. Gazette*, v. 10, pp. 119–128, 1920. (b) *James Stirling. A Sketch of his Life and Works along with his Scientific Correspondence.* Oxford, 1922, xii, 214 pp.+2 folding plates.

In the *Methodus* (prop. XXVIII, pp. 135–137) STIRLING gave the equivalent of the formula

$$\ln m! = \tfrac{1}{2}\ln(2\pi) + z\ln z - \frac{1}{2\cdot 12z} + \frac{7}{8\cdot 360z^3} - \frac{31}{32\cdot 1260z^5} + \frac{127}{128\cdot 1680z^7} - \cdots \tag{2}$$

where $z = m+\frac{1}{2}$. This is all that STIRLING gives. Nowhere in this connection does he consider m very large, or any formula such as (1). DE MOIVRE's *Miscellanea Analytica* was published in Jan. 1730. This did not then contain the "Supplementum" (21 p.) which was apparently written after the appearance of STIRLING's *Methodus*, and published with bound copies of his *Miscellanea* before the end of September 1730. DE MOIVRE here showed (pp. [8]–[9]) that

$$\ln m! = \tfrac{1}{2}\ln B + (m+\tfrac{1}{2})\ln m - m + \frac{1}{12m} - \frac{1}{360m^3} + \frac{1}{1260m^5} - \frac{1}{1680m^7} + \cdots \tag{3}$$

where

$$\tfrac{1}{2}\ln B = 1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680} - \cdots = \tfrac{1}{2}\ln(2\pi),$$

(the fact that $B = 2\pi$ being due to STIRLING), or

$$\log m! = \tfrac{1}{2}\log(2\pi m) + m\log m - m\log e + \frac{\log e}{12m} - \frac{\log e}{360m^3} + \frac{\log e}{1260m^5} - \frac{\log e}{1680m^7} + \cdots. \tag{4}$$

This may be written

$$\log m! = \log\left[(2\pi m)^{1/2} m^m e^{-m}\right] + \left[\frac{B_1}{1\cdot 2}\cdot\frac{1}{m} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{m^3} + \frac{B_3}{5\cdot 6}\cdot\frac{1}{m^5} - \cdots\right]\log e, \tag{5}$$

$B_1, B_2, \ldots$ denoting the BERNOULLI numbers. When m is very large, the relation (1) readily follows from (5). The values of the first 71 of the coefficients $(-1)^{n-1}B_n/[(2n-1)2n]$, for $n = 1(1)71$, were given in H. S. UHLER, "The coefficients of Stirling's series of log $\Gamma(x)$," Nat. Acad. Sci., *Proc.*, v. 28, 1942, pp. 59–62. On p. (22) of the *Miscellanea*, DE MOIVRE gives a table(15–19S) of log $m!$ $m = 10(10)900$, not without errors. DE MOIVRE returned to (1) in a private publication of Nov. 12, 1733: *Approximatio ad Summam Terminorum Binomii* $(a+b)^n$ *in Seriem expansi.* (7 p.), printed in facsimile in R. C. ARCHIBALD, "A rare pamphlet of Moivre and some of his discoveries," *Isis*, v. 8, pp. 671–676+7 plates, 1926. A somewhat elaborated English edition of this pamphlet is given in the second and third editions of DE MOIVRE's *The Doctrine of Chances* (1738, 1756). See also K. PEARSON (a) "Historical note on the origin of the normal curve of errors," (b) "James Bernoulli's

theorem," *Biometrika*, (a) v. 16, pp. 402–404, 1924; (b) v. 17, pp. 201–210, 1925. It is of special interest that DE MOIVRE tells us that the part of his 1733 publication into which we are here inquiring dates back to 1721 at least ("It is now a dozen years or more since I had found what follows"). Writing 23 years later as on p. 334 of the Appendix to the third edition of his *Doctrine of Chances* the genial De Moivre was evidently somewhat forgetful of the exact setting of his friend's relationship to one of his own discoveries.

PEARSON'S articles make clear that to DE MOIVRE in 1733 is also due: (a) the first treatment of the probability integral, and essentially of the normal curve (later given by LAPLACE in 1774 and 1778; and by GAUSS in the 19th century); (b) the theorem that the measure of accuracy depends on the inverse square root of the size of the sample, so often called BERNOULLI'S theorem, although wholly DE MOIVRE'S.

232. SMITH [150], "De Moivre on his formula", pp. 440–450, "Euler," pp. 450–454, ed. by R. C. ARCHIBALD. BRAUNMÜHL [131], v. 2, pp. 75–78. A. v. BRAUNMÜHL, "Zur Geschichte der Entstehung des sogenannten Moivrischen Satzes," *Bibl. Math.*, v. 2, 1901, pp. 97–102.

233. E. I. CARLYLE, "Taylor, Brook," *Dict. Nat. Biog.*, v. 55, 1898. B. TAYLOR, *Contemplatio Philosophica: a posthumous work. . . . To which is prefixed a life of the author by his grandson Sir William Young.* London, 1793, iv, 150 pp. H. BATEMAN, "The correspondence of Brook Taylor," *Bibl. Math.*, v. 7, pp. 367–371, 1907. One of the mss. "dated July 26, 1712 contains an enunciation of the theorem which bears his name." H. AUCHTER, *Brook Taylor der Mathematiker und Philosoph. Beiträge zur Wissenschaftsgeschichte der Zeit des Newton-Leibniz-Streites.* Diss. Marburg. Würzburg, 1937, iv, 112 pp.+portrait plate reproduced from previous item; extracts from Taylor's correspondence, pp. 72–97; "Contemplatio philosophica, an essay by Brook Taylor," pp. 98–110. A. PRINGSHEIM, "Zur Geschichte des Taylorschen Lehrsatzes," *Bibl. Math.*, s. 3, v. 1, 1900, pp. 433–479. Of TAYLOR'S *Methodus Incrementorum*, there was a second ed., or print, in 1717 and a facsimile print of this at Berlin, in 1862. F. J. FÉTIS, *Biographie Universelle de Musiciens*, Paris, v. 8, 1884, p. 194.

234. ARCHIBALD [57], B. Taylor, p. 19.

235. C. PLATTS, "Maclaurin, Colin," *Dict. Nat. Biog.*, v. 35, 1893. C. TWEEDIE, "A study of the life and writings of Colin Maclaurin," and "Notes on the life and works of Colin Maclaurin," *Math. Gazette*, v. 8, pp. 133–151+4 plates, 1915, and v. 9, pp. 303–305, 1919. H. W. TURNBULL, "Colin Maclaurin," *Amer. Math. Monthly*, 1947, v. 54, pp. 318–322. CAJORI [209], chap. VI, "Maclaurin's Treatise of Fluxions, 1742," pp. 181–189. C. TWEEDIE, "The 'Geometria Organica' of Colin Maclaurin: a historical and critical survey," R. Soc. Edinb., *Proc.*, v. 36, pp. 87–150, 1916.

236. "Un chef-d'oeuvre de Géométrie, qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux" (*Mém. de l'Acad. de Berlin*, 1773, p. 121). A reference may also be given to G. A. GIBSON, "Sketch of the history of mathematics in Scotland to the end of the 18th century," Edinb. Math. Soc., *Proc.*, s. 2, v. 1, pp. 79–84, 1928.

237. ARCHIBALD [69], "pedal curves," "cissoid," *etc.*

238. R. E. LANGER, "The life of Léonard Euler," *Scripta Math.*, v. 3, 1935, pp. 61–66, 131–138. BELL [53], "Analysis incarnate-Euler," pp. 139–152. SMITH [54]. R. FUETER, *Leonhard Euler.* (*Kurze Mathematiker-Biographien*, no. 3). *Z. d. Elemente d. Mathematik, Beiheft*, Basel, Jan. 1948, 24 pp. There are 3 illustrs. (2 portraits and a facsimile of a page of Euler's writing). L. G. DU PASQUIER, *Léonard Euler et ses Amis.* Paris, 1927, ix, 125 pp. R. WOLF, *Biographien zur Kulturgeschichte der Schweiz*, 4 v., Zürich, v. 4, 1862, pp. 87–134. *Festschrift zur Feier des 200. Geburtstages Eulers.* (*Abh. z. Gesch. d. math. Wissen.*, Heft 25). Leipzig, 1907, iv, 137 pp.+2 portraits. [Contents: G. VALENTIN, "Leonhard Euler in Berlin," pp. 1–20; A. KNESER, "Euler und die Variationsrechnung," pp. 21–60; F. MÜLLER, "Über bahnbrechende Arbeiten Leonhard Eulers aus der reinen Mathematik," pp. 61–116; E. LAMPE, "Zur Entstehung der Begriffe der Exponentialfunktion und der logarithmischen Funktion eines komplexen Arguments bei Leonhard Euler," pp. 117–137.] G. ENESTRÖM, *Verzeichnis der Schriften Leonhard Eulers.* Leipzig, 1910–1913, 388 pp. EULER'S works are now being published by the Society of Swiss Naturalists in *Opera Omnia* (33 v., 1911–1947, there are to be about 40 more v.).

SMITH [150], "Euler, proof that every integer is a sum of four squares," ed. by E. T. BELL, pp. 91–94 (see also [99, 101]; "Euler's use of the letter e to represent 2.718" ed. by F. CAJORI, pp. 95–98; "Euler on differential equations of the second order," ed. by F. CAJORI, pp. 638–643. U. G. MITCHELL & M. STRAIN, "The number e," *Osiris*, v. 1, 1936, pp. 476–496. D. H. LEHMER, "On the value of the Naperian base," *Amer. Jn. Math.*, v. 48, pp. 139–143, 1936; the value of e is here found to 707D to match the then supposedly accurate SHANKS value of π to 707D—see [201]. R. C. ARCHIBALD, "Euler's integrals and Euler's spiral—sometimes called Fresnel integrals and the clothoide or Cornu's spiral," *Amer. Math. Monthly*, v. 25, 1918, pp. 276–282. This spiral is the elastic curve in the following: L. EULER, "Elastic curves," transl. and annotated by W. A. OLDFATHER, C. A. ELLIS & D. M. BROWN, *Isis*, v. 20, pp. 72–160, 1933; this is a complete translation of "Additamentum I. De Curvis Elasticis," pp. 245–310 of EULER's Methodus inveniendi Lineas Curvas Maximi Minimive proprietate gaudentes. (Lausanne and Geneva, 1744). Everything in H. LISENBARTH's edition of a German transl. of the text, in OSTWALD's *Klassiker d. exakten Wissen.*, v. 175 (Leipzig, 1910), has been incorporated into this translation; and there is little else. ORE [7b], "Euler's theorem and its consequences," pp. 272–310.

238a. F. CAJORI, "Frederick the Great on mathematics and mathematicians," *Amer. Math. Monthly*, v. 34, 1927, pp. 122–130.

239. TODHUNTER [189], chap. XII, Euler, pp. 239–257; *etc.*

240. A. ENNEPER, *Elliptische Functionen. Theorie und Geschichte.* Second ed. by F. MÜLLER, Halle a. S., 1890, xx, 598 pp. More than a dozen papers of EULER are here analyzed.

241. ARCHIBALD [57], pp. 20–21.

242. CAJORI [163], v. 1–2, see indices.

243. F. LINDEMANN, "Ueber die Zahl π," *Mathem. Annalen*, v. 20, 1882, pp. 213–225.

244. H. S. UHLER, "On the numerical value of i^i" and R. C. ARCHIBALD, "Historical notes on the relation $e^{-\frac{1}{2}\pi} = i^i$," *Amer. Math. Monthly*, v. 28, 1921, pp. 114–121.

245. J. S. MACKAY, "An abstract of one of Euler's papers," Edinb. Math. Soc., *Proc.*, v. 4, 1886, pp. 51–55. See also SMITH [150], "Feuerbach on the theorem which bears his name," ed. by R. A. JOHNSON, pp. 339–345. And J. L. COOLIDGE, *A History of Geometrical Methods*, Oxford, 1940, xviii, 452 pp.; pp. 66, 147.

246. A. DEMORGAN, "Fourier, Joseph," *Penny Cycl.*, v. 10, 1838; and *Engl. Cycl.-Biog.*, v. 2, 1856. BELL [53], "Friends of an emperor, Monge and Fourier," pp. 183–203. BELL [5], pp. 292–294, etc. F. ARAGO, "Joseph Fourier" (read 1833) transl. into English, Smithsonian Institution, *Annual Report*, 1871, pp. 137–176; also in F. ARAGO, *Biographies of Distinguished Men*, Boston, 1859, v. 1, pp. 374–444. M. BÔCHER, "Historical summary," pp. 267–275, of W. E. BYERLY, *An Elementary Treatise on Fourier Series and Spherical, Cylindrical, and Ellipsoidal Harmonics.* Boston, 1893. See also the admirable survey, R. E. LANGER, *Fourier's Series, the Genesis and Evolution of a Theory.* (*Herbert Ellsworth Slaught Memorial Papers*, no. 1). Suppl. *Amer. Math. Monthly*, v. 54, 1947, vi, 86 pp. Of Fourier's great work *Théorie Analytique de la Chaleur*, Paris, 1822, there was an English translation by A. FREEMAN, *The Analytic Theory of Heat, translated with Notes.* Cambridge, Univ. Press, 1878, xxviii, 466 pp. The French edition filled the first of two v. of *Oeuvres de Fourier*, ed. by G. DARBOUX. Paris 1888–1890. These v. do not include the following work not completely published at the time of Fourier's death: *Analyse des Équations Déterminées. Première Partie* [no more publ.]. Paris 1831, xxiv, 258 pp. +1 folding plate. This v. deals with topics which had interested him from early manhood, and the noted theorem (perfected by STURM in 1829, publ. in 1835) concerning the number of roots of an algebraic equation in a given interval, is certainly due to him both because it was taught to his pupils in the École Polytechnique in 1796, 1797, and 1803, and because he first printed the theorem and its proof in 1820. (*Oeuvres*, v. 2, pp. 291–309). Many writers (including ARAGO, above) were wholly incorrect in imagining that F. D. BUDAN gave the theorem in his book of 1806. BUDAN did finally give a complete demonstration in 1822. P. E. B. JOURDAIN, "Note on Fourier's influence on the conceptions of mathematics," Intern. Congress Mathems., Cambridge 1912, *Proc.*, 1912, v. 2, pp. 526–527. N. NIELSEN, *Géomètres Français sous la Révolution.* Copenhagen, 1929, viii, 251 pp. "Fourier," pp. 88–96; "Budan," pp. 38–39, with ac-

curate details regarding Fourier's theorem. The 80 sketches in this volume include also the following: LAGRANGE, pp. 136–152, LAPLACE, pp. 157–163, LEGENDRE, pp. 166–174, MONGE, pp. 182–190, J. R. ARGAND (1768–1822), pp. 6–9, J. C. DE BORDA (1733–1799), pp. 19–24, J. C. CALLET (1744–1799) pp. 40–44, J. B. J. DELAMBRE (1749–1822), pp. 70–72, J. J. LE F. DE LA LANDE (1732–1807), pp. 43–45, J. E. MONTUCLA (1725–1799), pp. 190–192. Concerning the last six we shall add some notes:

ARGAND—We referred to WALLIS's attempt to represent imaginary quantities graphically. ARGAND gave our modern method in 1806. He was, however, anticipated by CASPAR WESSEL (1745–1818) whose memoir, presented to the Danish Academy in 1797, was published in 1799. For a good history of geometric representation of imaginary quantities, see W. W. BEMAN, "A chapter in the history of mathematics," Amer. Assoc. Adv. Sci., *Proc.*, v. 46, 1897, pp. 33–50. An English translation of ARGAND's paper is: *Imaginary Quantities: their Geometrical Interpretation. Translated from the French of M. Argand by* A. S. HARDY. New York, 1881, 135 p.

BORDA—See the sketch of BORDA by A. DEMORGAN, in *Penny Cycl.*, v. 5, 1836, and *Engl. Cycl.-Biog.*, v. 1, 1856. BORDA, LAGRANGE, LAPLACE, MONGE & CONDORCET, in a report, made in 1791 to the Academy of Sciences, Paris, recommended the decimal division of the quadrant of the circle (100^g, $1^g = 100'$, $1' = 100''$). This subdivision is found in CALLET's tables, and, 1801, in BORDA's posthumous: *Tables Trigonométriques Décimales, ou Tables des Logarithmes*, with revision and an explanation by DELAMBRE (iv, 637 pp.). Through recommendation of the above committee the meter was in 1799 adopted as the standard of length.

CALLET—was the author of a famous volume: *Tables portatives de Logarithmes*, 1795, cxviii, 680 pp., later appearing in scores of editions and impressions.

DELAMBRE—see also the excellent sketch by A. DEMORGAN, in *Penny Cycl.*, v. 8, 1937, and *Engl. Cycl.-Biog.*, v. 2, 1856. D. was chosen an associate of almost every learned body in Europe. He was the author of 6 great v. of History of Astronomy (1817–1827), and many other v. And also the author of what are known as "GAUSS's analogies" [272] in trigonometry; see I. TODHUNTER, "Note on the history of certain formulae in spherical trigonometry," *Phil. Mag.*, s. 4, v. 45, 1873, pp. 98–100.

LALANDE—see also A. DEMORGAN, *Penny Cycl.*, v. 13, 1939, and *Engl. Cycl.-Biog.*, v. 3, 1856; and ARCHIBALD [163], pp. 43–45. Director of the Paris Observatory (1768–1807). Author of a v. of mathematical tables which went through many editions down into the 20th century (ARCHIBALD [163], pp. 43–45); of Navigation its history theory and practice (1793); and of the fourth v. of MONTUCLA's History (1802).

MONTUCLA—author of two noteworthy treatises on the history of mathematics. One was on the quadrature of the circle (1754, second ed. by LACROIX, 1831); the other and larger work was the first modern history of mathematics, and it may be called a classic (2 v. Paris, 1758; second edition 4 v. 1799–1802. The third v. was partly printed when M. died; the rest of it (after p. 336) was seen through the press by LALANDE who prepared the fourth v., mainly on the history of astronomy.) See G. SARTON, "Montucla (1725–1799): his life and works," *Osiris*, v. 1, 1936, pp. 519–567. Also D. E. SMITH, "Among my autographs": "The threatened loss of the second edition of Montucla's *History of Mathematics*"; "Montucla's closing years," *Amer. Math. Monthly*, v. 28, 1928, pp. 207–208; v. 29, 1929, pp. 253–255.

247. M. GODEFROY, *La Fonction Gamma: Théorie, Histoire, Bibliographie.* Diss. Paris. Paris, 1901, pp. 1–6, *etc.*, EULER's Beta Function, $\int_0^1 x^{p-1}(1-x)^{q-1}dx$, and the more general form, $\int_0^1 x^{p-1}dx/(1-x^n)^{(n-q)/n}$, were studied by EULER in papers of 1738, 1750, 1761, and 1789; see, ENESTRÖM [238].

248. $$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}=\gamma+\ln x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_2}{4x^4}-\frac{B_3}{6x^6}+\cdots.$$

J. W. L. GLAISHER, "On the history of Euler's constant," *Mess. Moth.*, v. 1, pp. 25–30, 1871. The constant γ was calculated to 263D by J. C. ADAMS, B.A.A.S., *Report, 1877*, p. 15 and R. Soc. London, *Proc.*, v. 27, 1878, pp. 93–94.

249. G. SARTON, "Lagrange's personality (1736–1813)," Amer. Phil. Soc., *Proc.*, v. 88, 1944, pp. 457–496+portrait frontispiece. BELL [53], "A lofty pyramid, Lagrange," pp. 153–171. SMITH [54], NIELSEN [246], pp. 136–152. TURNBULL [51], "Maclaurin and Lagrange," pp. 101–107. O. J. LODGE, *Pioneers of Science*, London, 1919; lecture XI: "Lagrange and Laplace—the stability of the solar system, and the nebular hypothesis," pp. 254–271. "Bicentenary of Lagrange (1736–1813)," *Nature*, v. 137, 1936, pp. 141–142. G. LORIA, (a) "Nel secondo centenario della nascita di G. L. Lagrange 1736–1936," *Isis*, v. 28, 1938, pp. 366–375, with interesting illustrations. (b) "G. L. Lagrange nella vita e nelle opere," *Annali di Matem.*, s. 3, v. 20, 1913, pp. IX–LII. L. G. SIMONS, "The influence of French mathematicians at the end of the eighteenth century upon the teaching of mathematics in American Colleges," *Isis*, v. 15, 1931, pp. 104–106, 108, 109, 119.

250. *Engl. Cycl.-Biog.*, v. 3, 1856, "Lambert, John Henry." J. H. LAMBERT, *Opera Mathematica, Commentationes Arithmeticae Algebraicae Analyticae*, ed. ANDREAS SPEISER, V. 1–2, Zürich, 1946–1948, xxxii, 348, xxx, 324 pp.; the introductory essays of v. 1–2 by the editor, are synopses of interest. Pp. 1–15 of v. 1 are occupied by "Éloge de M. LAMBERT" by J. H. S. FORMEY, reprinted from *Nouv. Mém. de l'Acad. r. de Berlin 1778, Histoire*, pp. 72–91. R. WOLF, *Biographien zur Kulturgeschichte der Schweiz.* 4 v., Zürich, 1858–1862; v. 3, 1860, pp. 317–356. A sumptuous work dealing primarily with LAMBERT's writings on perspective but listing all of his published and unpublished writings is *Johann Heinrich Lambert Schriften zur Perspektive. Herausgegeben und eingeleitet von* MAX STECK. Berlin, 1943, xvii, 496 pp.+21 plates, folio format.

251. *Engl. Cycl.-Biog.*, "Wallace, William," v. 5, 1857. In LEYBOURN's *Math. Repository*, n.s., no. IV, Oct. 28, 1797, Wallace proposed the problem: "If three straight lines touch a parabola a circle described through their intersections shall pass through the focus of the parabola. Required the Demonstration." In his proof (no. VI, 1798, II, pp. 54–55) he proves that the feet of the perpendiculars from the focus of the parabola on the three tangents all lie on the tangent at the vertex of the parabola. Then in no. VII, 1799, II, p. 111 the WALLACE Line theorem emerges. PONCELET, who doubtless never saw WALLACE's articles, treats the theorems in the same way, see his *Propriétés Projectives*, 1822, §§466–468. In the latter paragraph is a generalization of the theorem, namely: that the perpendiculars on the sides may be replaced by obliques making (in cyclical order) equal angles with the sides. The same generalization was also made by STEINER in GERGONNE's *Annales*, v. 19, 1828, pp. 37–64. STEINER remarks that the straight lines obtained in varying the angles of the obliques will envelop a parabola, whose focus is the point from which the obliques are drawn. That the envelope of all the WALLACE Lines of a circle is a three-cusped hypocycloid was shown by W. K. CLIFFORD in 1865, *Educ. Times Reprint*, v. 3, pp. 81–82. The number of theorems connected with the WALLACE Line is very large. Some references may be found in J. S. MACKAY, "The Wallace line and the Wallace point," Edinb. Math. Soc., *Proc.*, v. 9, 1891, pp. 83–91; and in the remarkable *Frère* GABRIEL MARIE, *Exercices de Géométrie*, fifth ed., Paris, 1912, §764–767. See also J. S. MACKAY, "Bibliography of the envelope of Wallace line the three-cusped hypocycloid," Edinb. Math. Soc., *Proc.*, v. 23, 1905, pp. 80–88; and a bibliography of the extension of WALLACE's Line to space by O. DEGEL, in *L'Intermédiaire des Mathém.*, v. 18, 1911, pp. 254–255.

252. F. RUDIO, *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung. Deutsch herausgegeben und mit einer Übersicht über die Geschichte des Problems von der Quadratur des Zirkels . . . versehen von* F. RUDIO. Leipzig, 1892, pp. 54–58, 133–155. Compare A. PRINGSHEIM, "Ueber die ersten Beweise der Irrationalität von e und π," Bayerische Akad. d. Wissen., *Sitzungsb., math.-phys. Classe*, 1898, pp. 325–337. It is here pointed out that LAMBERT did not know of EULER's work 30 years earlier; also that LAMBERT's proof of the irrationality of π is more rigorous than the later one of LEGENDRE; that LAMBERT here gave the first, and for a long time the only, example of the rigorous development of a particular function into a converging continued fraction. See also J. POPKEN, "On Lambert's proof for the irrationality of π," Akad. v. Wetens., Amsterdam, *Proc.*, v. 43, 1940, pp. 712–714. The simplest proof yet found for the irrationality of π seems to be that of I. NIVEN, Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 509. HERMITE's proof that π and π^2 were irrational is of interest both in relation to the proof of LAMBERT and as containing the germ of the later proof of the transcendence of e and π (*Jn. f. d. reine u.*

angew. Math., v. 76, 1873, pp. 342–344); see also *Oeuvres de Charles Hermite*, v. 3, 1912, pp. 146–149. A simple proof of the irrationality of e was given in 1815 by FOURIER. D. E. SMITH, "The history of the transcendence of π," in J. W. A. YOUNG, *Monographs on Topics of Modern Mathematics*, New York, 1911, pp. 387–416.

253. G. LORIA, "J. Liouville and his work," *Scripta Math.*, v. 4, 1936, pp. 147–154, 257–262, 301–306, and portrait plate.

254. BELL [53], "The man, not the method—Hermite," pp. 448–465. G. PRASAD, *Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Works*. Benares, India, v. 1, 1933 (Gauss, Cauchy, Abel, Jacobi, Weierstrass, Riemann), xv, 347 pp.+6 portraits; v. 2, 1934 (Cayley, Hermite, pp. 34–59, Brioschi, Kronecker, Cremona, Darboux, Cantor, Mittag-Leffler, Klein, Poincaré), xviii, 324 pp.+10 portraits. G. DARBOUX, *Éloges académiques et Discours*, Paris, 1912, pp. 116–172. M. NOETHER, *Math. Annalen*, v. 55, 1901, pp. 337–385. LOREY [276], JACOBI, pp. 43–44.

255. S. GÜNTHER, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen*. Halle a S., 1881, x, 440 pp.

256. C. H. DEETZ & O. S. ADAMS, *Elements of Map Projection with Applications to Map and Chart Construction*. Fifth ed., rev., Washington, 1945, 256 pp.+14 plates.

257. BELL [53], "Friends of an Emperor—Monge and Fourier," pp. 183–205. A. DEMORGAN, *Engl. Cycl.-Biog.*, v. 4, 1857, "Monge, Gaspard." D. E. SMITH, "Gaspard Monge, politician," *Scripta Math.*, v. 1, pp. 111–122, 1932. D. E. SMITH, "Among my autographs: Monge and the American Colonies" and "Monge the lesser," *Amer. Math. Monthly*, v. 28, 1921, pp. 166, 208, 209. SIMONS [249], pp. 115, 119. NIELSEN [244], "Monge," pp. 182–190. L. DELAUNAY, *Monge, Fondateur de l'École Polytechnique*. Paris, 1933, 280 pp. D. F. J. ARAGO, "Biographie de Gaspard Monge," *Mémoires de l'Acad. d. Sci.*, Paris, v. 24, 1853, 157 pp.; also in ARAGO, *Notices Biographiques*, Paris, v. 2, 1853, pp. 427–592. C. DUPIN, *Essai Historique sur les Services et les Travaux Scientifiques de Gaspard Monge*. Paris, 1819, viii, 316 pp. For Descriptive Geometry used by PTOLEMY, see NEUGEBAUER [7a], and HEATH [22], *History*, v. 2, pp. 286–292.

258. D. J. STRUIK, "Outline of a history of differential geometry," *Isis*, v. 19, 1933, pp. 92–120 (1. "The time before Leibniz," pp. 92–96; 2. "The first systematic contributions," pp. 96–99; 3. "The eighteenth century," pp. 99–110; 4. "Monge and the École Polytechnique," pp. 110–113; 5. "Monge's pupils," pp. 113–120), and v. 20, pp. 161–191, 1933 (6. "Gauss," pp. 161–167; 7. "The French school of the forties," pp. 167–172; 8. "Riemann," pp. 173–177; 9. "The beginning of modern times," pp. 177–184; 10. "Differential geometry from 1870 to 1900," pp. 184–189; 11. "Sources," pp. 189–191). See also J. L. COOLIDGE, *A History of Geometrical Methods*. Oxford, 1940; "Differential geometry," pp. 318–421.

259. F. CAJORI, "Plans for a history of mathematics in the nineteenth century," *Science*, n.s., v. 48, 1948, pp. 279–284; address as retiring president of the Math. Assoc. Amer.

260. A. DEMORGAN, "Laplace," *Penny Cycl.*, v. 13, 1839, and *Engl. Cycl.-Biog.*, v. 3, 1856, cols. 800–805. BELL [53], "From peasant to snob—Laplace," pp. 172–182. K. PEARSON, "Laplace"; G. A. SIMON, "Les origines de Laplace: sa généalogie—ses études," *Biometrika*, v. 21, 1930, pp. 202–216; pp. 217–230+1 plate. D. F. J. ARAGO, "Eulogy on Laplace," transl. into English by B. POWELL, Smithsonian Inst., *Report, 1874*, pp. 129–168. H. MACPHERSON, *Makers of Astronomy*, Oxford, 1933; "Laplace," pp. 88–93. LENARD [202], pp. 218–223+portrait plate. SMITH [54]. NIELSEN [246], "Laplace," pp. 157–163. H. ANDOYER, "*L'Oeuvre Scientifique de Laplace*. Paris, 1922, 162 pp. STRUIK [6], pp. 192–199. E. S. SMITH, "The tomb of Laplace," *Nature*, v. 119, 1927, pp. 493–494; quotation (p. 494): "On 11 Dec. 1925, a fire completely destroyed the Chateau de Mailloc, and with this were lost all the papers and personal relics of Laplace."; see also *Nature*, v. 4, 1871, p. 108. E. ESCLANGON, "Hommage à Laplace, inauguration de sa statue, à Beaumont-en-Auge le 3 Juillet 1932," *L'Astronomie*, v. 46, 1932, pp. 402–404. *Oeuvres Complètes de Laplace*. 14 v., Paris, 1878–1912.

261. TODHUNTER [189], "Laplace," pp. 464–613, *etc.* E. C. MOLINA, "The theory of probability: some comments on Laplace's *Théorie Analytique* [*des Probabilités*]," Amer. Math. Soc., *Bull.*, v. 36, 1930, pp. 369–392. The fundamental parts of LAPLACE's *Théorie* are given by A.

DEMORGAN in his article "Theory of Probabilities" in *Encycl. Metropolitana*, 1837. LAPLACE's "Essai Philosophique sur les Probabilités" was first printed as an introduction to the second edition of the *Théorie*, 1814. An English translation from the sixth French edition of the *Essai* (*A Philosophical Essay on Probabilities*) was published at New York, in 1902, 196 pp. SMITH [150], "Laplace on the probability of the errors in the mean results of a great number of observations and on the most advantageous mean result," ed. by J. L. C. A. GŸS, pp. 588–604.

262. R. C. ARCHIBALD, "Bowditch, Nathaniel," *Dict. Amer. Biog.*, v. 2, 1929; also *A Catalogue of a Special Exhibition of Manuscripts, Books, Portraits and Personal Relics of Nathaniel Bowditch (1773–1838)*. Salem, Mass., 1937, iv, 40 pp.+7 plates. ("Nathaniel Bowditch," by H. BOWDITCH, pp. 1–6; "The scientific achievements of Nathaniel Bowditch," by R. C. A., pp. 7–16). A. STANFORD, *Navigator. The Story of Nathaniel Bowditch*, New York, 1937, xii, 308 pp. R. E. BERRY, *Yankee Stargazer: the Life of Nathaniel Bowditch*, London, and New York, 1941, 234 pp.

263. A. DEMORGAN, "Legendre," *Penny Cycl.*, v. 13, 1839, and *Engl. Cycl.-Biog.*, v. 3, 1856. J. W. L. GLAISHER, "Legendre," *Encycl. Brit.* eleventh ed., Cambridge, v. 16, 1911 [G. errs in attributing to Sir DAVID BREWSTER the transl. into English of LEGENDRE's geometry]. L. ÉLIE DE BEAUMONT, "Éloge historique de Adrien Marie Legendre," Acad. d. Sci., Paris, Mémoires, v. 32, 1864, pp. xxxvii-xciv, 56 pp.; Engl. transl. by C. A. ALEXANDER, Smithsonian Inst., *Report for 1867*, pp. 137–157. NIELSEN [246], "Legendre," pp. 166–174. ARCHIBALD [163], pp. 45–49. C. DORIS HELLMAN, "Legendre and the French reform of weights and measures," *Osiris*, v. 1, 1936, pp. 314–340. I. TODHUNTER, *History of the Mathematical Theories of Attraction and the Figure of the Earth*. London 1873; in v. 2, chapters 20, 22, 24, 25, pp. 20–139, contain complete accounts of LEGENDRE's memoirs on attractions of ellipsoids. I. TODHUNTER, *A History of the Progress of the Calculus of Variations during the Nineteenth Century*, Cambridge and London, 1861, pp. 229–233, 489. L. E. DICKSON, *History of the Theory of Numbers*. Washington, v. 1–3, 1919, 1920, 1927; also New York, Stechert ed., 1935; see various references to LEGENDRE in indices. SMITH [150], "Legendre's law of quadratic reciprocity," ed. by D. H. LEHMER, pp. 112 f. D. E. SMITH, "Among my autographs: Legendre and Cauchy sponsor Abel," *Amer. Math. Monthly*, v. 29, 1922, pp. 394–395.

264. Legendre's work in Elliptic Functions is summarized in ENNEPER [240], index, pp. 586–587. His final volumes appeared almost simultaneously with the complete theory of elliptic functions by the great Norwegian mathematical genius **NIELS HENRIK ABEL**, which Legendre recognized as a great advance over his own. See BELL [53], "Genius and poverty—Abel," pp. 307–326. PRASAD [254], v. 1, pp. 111–165+portrait. SMITH [150], "Abel on the quintic equation" ed. by W. H. LANGDON & O. ORE, pp. 261–266. [It is impossible to solve the general equation of the fifth, or higher, degree, in terms of radicals.]; "Abel on the continuity of functions defined by power series," ed. by A. A. BENNETT, pp. 286–291; "Abel on integral equations," ed. by J. D. TAMARKIN, pp. 656–662. G. MITTAG-LEFFLER, "Niels Henrik Abel," *La Revue du Mois*, 1907, pp. 34–38. J. BERTRAND, *Éloges Académiques*, Paris, 1902; "N. H. A. Tableau de sa vie et de son action scientifique," pp. 313–328.

LEGENDRE was similarly impressed with the work of **C. G. J. JACOBI** which appeared somewhat later (1829). BELL [53], "The great algorist—Jacobi," pp. 327–339. PRASAD [254], v. 1, pp. 165–219+portrait plate. SMITH [54]. STRUIK [6] pp. 227–232. "Correspondance mathématique entre Legendre et Jacobi," *Jn. f. d. reine u. angew. Math.*, v. 80, 1875, pp. 205–279; also in JACOBI, *Gesammelte Werke*, v. 1, Berlin, 1881, pp. 385–461. ENNEPER [240], pp. 581–583. MUIR [219], many references in indices and contents to "Jacobians," *etc.* I. TODHUNTER, *History of the Progress of the Calculus of Variations during the Nineteenth Century*. Cambridge, 1861; "Jacobi," pp. 243–253. "Commentators on Jacobi," pp. 254–332. L. KOENIGSBERGER, *Carl Gustav Jacob Jacobi. . . .* Leipzig, 1904, xviii, 554 pp.+1 plate.

265. M. MERRIMAN, "List of writings related to the method of least squares with historical and critical notes," Connecticut Acad., *Trans.*, v. 4, 1877, pp. 160–173. For a transl. of a portion of LEGENDRE'S "On a method of least squares," ed. by H. A. RUGER & H. M. WALKER, see SMITH [150], pp. 576–579.

266. The first English translation of LEGENDRE's geometry was by JOHN FARRAR, of Harvard

University, Cambridge, Mass., 1819. The next English translation, together with an introductory chapter on proportion, was by THOMAS CARLYLE, Edinburgh, 1824. There were 39 American editions (6 of FARRAR and 33 of CARLYLE) 1819–1880, and the work led to the practical abandonment in the United States of EUCLID'S *Elements*. See SIMONS [249], pp. 107–115; and L. C. KARPINSKI, *Bibliography of Mathematical Works Printed in America through 1850*. Ann Arbor, 1940, pp. 11, 228–229, 292–293, 449, 459. *The Thirteen Books of Euclid's Elements Translated from the Text of Heiberg* with Introd. and Commentary by T. L. HEATH, v. 1, Cambridge, 1908, pp. 112, 169, 213–219; second ed. rev. with additions, 1926. BONOLA [78], pp. 55–60, *etc.* In a copy of THOS. SIMPSON, *The Doctrine and Application of Fluxions*, 1776, in the Library of Brown University, is the signature: "Thomas Carlyle, Studt. Edinb. 1814." This is the year when Carlyle (19 years of age) was mathematical teacher at Annan.

267. BELL [53], "The Copernicus of geometry—Lobatchewsky," pp. 307–326, G. B. HALSTED, "Lobachevsky," *Amer. Math. Monthly*, v. 2, 1895, pp. 137–139. A. V. VASILIEV, *Nicolai Ivanovich Lobachevsky, Address pronounced at the commemorative Meeting of the Univ. of Kazan, Oct. 22, 1893*, transl. by G. B. HALSTED, Austin, Texas, 1894, viii, 40 pp. LOBACHEVSKY'S noneuclidean geometry writings included: (1) *Ueber die Principien der Geometrie* (Kazan, 1829–1830); (2) *Geometrische Untersuchungen zur Theorie der Parallellinien*, Berlin, 1840, French transl. by HOÜEL, 1866 and English transl. by HALSTED, *Geometrical Researches on the Theory of Parallels*, Austin, Texas, 1891, 50 pp.; new ed. with portrait and bibl., Chicago, 1914, 50 pp. L. published also (3) *Pangéométrie ou Précis de Géométrie fondée sur une Théorie générale et rigoureuse des Parallèles*, Kazan, 1855. BONOLA [78], pp. 84–96, *etc.* In 1944 the Academy of Sciences, Moscow-Leningrad. pubi. a 348-page biog. of LOBACHEVSKY by V. F. KAGAN.

268. E. T. BELL, "Father and son Wolfgang and Johann Bolyai," *Scripta Math.*, v. 5, 1938, pp. 37–44, 95–100. G. B. HALSTED, (a) "Bolyai Farkas," and (b) "Bolyai Janos," *Amer. Math. Monthly*, (a) v. 3, 1896, pp. 1–5; (b) v. 5, 1898, pp. 35–38. BONOLA [78], pp. 96–113, *etc.* JÁNOS BOLYAI'S Latin Appendix of 1832 was translated into English: *The Science Absolute of Space, . . . independent of the Truth or Falsity of Euclid's Axiom XI (which can never be decided a priori). . . .* Austin, Texas, fourth ed., 1896, 71 pp. P. G. STÄCKEL, *Wolfgang und Johann Bolyai, geometrische Untersuchungen*. Leipzig, 1913, 2 v., viii, 274, xii, 282 pp. *Briefwechsel zwischen Carl Friedrich Gauss und Wolfgang Bolyai*, ed. by F. SCHMIDT u. P. G. STÄCKEL, Leipzig, 1899, 220 pp.+2 plates+14 facsimiles.

269. D. M. Y. SOMMERVILLE, *Bibliography of Non-Euclidean Geometry, including the Theory of Parallels, the Foundations of Geometry, and Space of n Dimensions*. London, 1911, xii, 404 pp.

270. BELL [53], "The prince of mathematicians—Gauss," pp. 218–269. PRASAD [254], v. 1, pp. vii–viii, 1–67+portrait frontispiece. LENARD [202], pp. 240–247. *Carl Friedrich Gauss, Inaugural Lecture on Astronomy and Papers on the Foundation of Mathematics*. Transl. and ed. by G. W. DUNNINGTON, Baton Rouge, La., 1937, xi, 91 pp.; "The early life of C. F. G." pp. 1–32. G. W. DUNNINGTON, "The historical significance of C. F. G. in mathematics and some aspects of his work," *Math. News Letter*, v. 8, 1934, pp. 175–179. F. CAJORI, (a) "C. F. G. and his children," *Science*, n.s., v. 9, 1899, pp. 697–704; (b) "Gauss and his American descendants," *Pop. Sci. Mo.*, v. 81, 1912, pp. 105–114. H. MACK, *Carl Friedrich Gauss und die Seinen. Festschrift zu seinem 150. Geburtstage*. Braunschweig, 1927, 12+150 pp.+12 plates; one finds here details concerning GAUSS'S sons who settled in the United States, and their descendants. SMITH [150], "Gauss on the congruence of numbers," ed. R. G. ARCHIBALD, pp. 107–111; "Gauss, third proof of the law of quadratic reciprocity," ed. D. H. LEHMER, pp. 112–118; "Gauss, second proof of the fundamental theorem of algebra," ed. C. R. ADAMS, pp. 292–306; "Gauss, on conformal representation," ed. H. P. EVANS, pp. 463–475. BONOLA [78], pp. 64–75, *etc.* STRUIK [258], pp. 161–167. P. E. B. JOURDAIN, "The theory of functions with Cauchy and Gauss," *Bibl. Math.*, v. 6, pp. 190–207, 1905. A. PRINGSHEIM, "Kritisch-historische Bemerkungen zur Funktionentheorie. V–VI. Über einen Gaussischen Beweis der Irrationalität von tang x bei rationalem x," Akad. d. Wissen., Munich, *Sitzb., math.-naturw. Abt.*, 1932, pp. 193–200, 1933, pp. 61–70. J. MUNRO, *Heroes of the Telegraph*, London, 1891, pp. 269–272. R. W. POHL, "Zur Jahrhundertfeier des elektromagnetischen Telegraphen von Gauss

und Weber . . . ", Gesell. d. Wissen. zu Göttingen, *Nach., Jahresb.*, 1933–34, pp. 48–56. F. KLEIN, *Vorlesungen über die Entwicklung der Math. im 19. Jahrhundert*, part 1. Berlin, 1926, pp. 6–60, etc.; see also part 2, 1927, index. F. KLEIN, "Gauss's wissenschaftliches Tagebuch, 1796–1814" Göttingen Gesell. d. Wissen., *Festschrift*, Berlin, 1901, pp. 1–44 with portr. and facs. *Materialien für eine wissenschaftliche Biographie von G.* Leipzig, 8 parts, 1911–1920. [1. P. BACHMANN, *Über Gauss' zahlentheoretische Arbeiten*, 1911, iv, 54 pp. 2–3. GAUSS, *Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799*, and L. SCHLESINGER, *Über Gauss' Arbeiten zur Funktionentheorie*. 1912, iv, 34, 143 pp. 4–5. A. GALLE, *C. F. Gauss als Zahlenrechner*, and P. STÄCKEL, *C. F. Gauss als Geometer*, 1918, iv, 142 pp. 6. P. MAENNCHEN, *Die Wechselwirkung zwischen Zahlenrechnen und Zahlentheorie bei C. F. Gauss*, 1918, iv, 47 pp. 7. M. BRENDEL, *Über die astronomischen Arbeiten von Gauss, . . . theoretische Astronomie*. 1919, iv, 106 pp. 8. A. FRAENKEL, *Zahlbegriff und Algebra bei Gauss*, mit einem Anhang von A. OSTROWSKI *zum ersten und vierten Gausschen Beweise des Fundamentalsatzes der Algebra*. 1920, ii, 59 pp. L. BIEBERBACH, *C. F. Gauss, ein deutsches Gelehrtenleben*. Berlin, 1938, 178 pp. Of Gauss's *Werke*, 12 v., in 14, have been publ. 1863–1933.

271. SMITH [150], "Gauss on the division of the circle into n equal parts," ed. J. S. TURNER, pp. 348–350. F. KLEIN, *Famous Problems of Elementary Geometry*, transl. by W. W. BEMAN & D. E. SMITH. Second ed. rev. and enl. with notes by R. C. ARCHIBALD, New York, 1930; "Gaussian polygons," pp. 81–85. J. PIERPONT, "On an undemonstrated theorem of the *Disquisitiones Arithmeticae*," Amer. Math. Soc., *Bull.*, v. 2, 1895, pp. 77–83. R. C. ARCHIBALD, G. W. DUNNINGTON, & H. GEPPERT, "Gauss's *Disquisitiones Arithmeticae* and the French Academy of Sciences," *etc., Scripta Math.*, v. 3, 1935, pp. 98, 193–196, 285–286, 356–358; v. 4, 1936, p. 106; and *Nat. Math. Mag.*, v. 9, 1935, pp. 187–192.

272. An American edition of one of GAUSS's famous astronomical works was: *Theory of the Motion of the Heavenly Bodies moving about the Sun in Conic Sections; a translation of Gauss's "Theoria Motus" with an Appendix*. By C. H. DAVIS, Cambridge, Mass., 1857, xvii, 326, 42 pp.+7 plates. In this work are found four formulae of spherical trigonometry, often called "GAUSS's analogies," although discovered considerably earlier by DELAMBRE [246].

273. JEAN LE ROND, dit D'ALEMBERT (1717–1783) mathematician, philosopher, author of many books and articles including *Opuscules Mathématiques* (8 v., Paris 1776–1780), *Encyclopédie Méthodique. Mathématiques.* (5 v., 1784–97, in collaboration with others), *Mélanges de Littérature, d'Histoire et de Philosophie* (5 v., new ed., Leyden, 1782–83), and *Éléments de Musique théorique et pratique suivant les Principes de M. Rameau éclaircis, développées et simplifiés* (Paris, 1752; fourth ed., Lyons, 1779). There is a very interesting sketch of D'Alembert by A. DEMORGAN, in *Penny Cycl.*, v. 1, 1833, and *Engl. Cycl.-Biog.*, "ALEMBERT," v. 1, 1856. L. LEMOINE, *Dictionnaire de Biographie Française*, ed. by J. BALTEAU, M. BARROUX, & M. PRÉVOST, Paris, v. 1, 1933, cols. 1397–1416; a long list of portraits of D'ALEMBERT and a bibliography of his life, cols. 1414–1416. J. L. F. BERTRAND, *Éloges Académiques*, nouv. série, Paris, 1902, "D'Alembert et Lagrange" pp. 291–311. J. L. F. BERTRAND, *Les Grands Écrivains: D'Alembert*. Paris, 1889, 206 pp. ARCHIBALD [57], pp. 19–21. STRUIK [7], pp. 182–184. SMITH [4], v. 1, pp. 479–480.

274. ARCHIBALD [163], pp. 21–22. E. W. SCRIPTURE, "Arithmetical prodigies," *Amer. Jn. Psych.*, v. 4, 1891, pp. 18–20, 40–41, etc. F. D. MITCHELL, "Mathematical prodigies," *Amer. Jn. Psych.*, v. 18, 1907, pp. 75–77, etc. R. C. ARCHIBALD, "Mathematicians, and poetry and drama," *Science*, n.s., v. 89, 1939, p. 46.

275. BELL [53], "The day of glory—Poncelet," pp. 206–217. *Encycl. Brit.*, eleventh ed., v. 22. CAJORI [2], pp. 287–288. SMITH [150], "Poncelet on projective geometry" ed. by V. SANFORD, pp. 315–323; "Brianchon and Poncelet on the nine-point circle theorem," ed. by M. M. SLOTNICK, pp. 337–338. J. L. F. BERTRAND, "Éloge historique de J. V. P.," Acad. d. Sci., Paris, *Mémoires*, v. 41, pt. 2, 1879, pp. i–xxv and *Éloges Académiques*, 1890, pp. 105–129. COOLIDGE [245], pp. 92–95. E. KÖTTER, *Die Entwickelung der synthetischen Geometrie*. Leipzig, 1901; "Poncelet," very numerous references in the index.

276. "Steiner"; *Encycl. Brit.*, eleventh ed., v. 25, 1911. CAJORI [2], pp. 290–292. L. KOLLROS,

Jakob Steiner. (*Kurze Mathematiker-Biographien*, no. 2). *Suppl. Revue de Mathématiques Élémentaires*, Basel, Dec. 1947, 24 pp. There are 3 illustrs. (2 portraits and a facsimile of a page of Steiner's Ms.). COOLIDGE [245], pp. 96–97. KÖTTER [275], many references in the index. J. LANGE, *Jacob Steiners Lebensjahre in Berlin, 1821–1863, nach seinen Personalakten dargestellt.* Berlin, 1899, 70 pp., portr. E. LAMPE, "Zur Biographie von J. S.," *Bibl. Math.*, s. 3, v. 1, pp. 129–141, 1900. W. LOREY, *Das Studium der Mathematik an den deutschen Universitäten seit Anfang des 19. Jahrhunderts*, Leipzig, 1916, Steiner, pp. 46–49, *etc.* J. STEINER, *Gesammelte Werke*, ed. K. WEIERSTRASS, Berlin 2, v., 1881–1882.

In 1803 G. F. MALFATTI of the Univ. of Ferrara proposed the problem to cut three cylindrical holes out of a triangular prism in such a way that the cylinders and the prism have the same altitude and that the volumes of the cylinders be a maximum. This problem was reduced to another, now generally known as MALFATTI's problem: to inscribe three circles in a triangle so that each circle is tangent to two sides of the triangle and to the other two circles. MALFATTI gave an analytical solution but STEINER published without proof a simple construction by synthetic geometry, remarking that there were 32 solutions, and generalized the problem by replacing the three lines by three circles, and solved the analogous generalized problem for three dimensions: To determine the three sections of a surface of the second order, each of them touching the other two and also two of three given sections of the surface of the second order. This general problem was solved analytically by CAYLEY and CLEBSCH, for example, with the aid of the addition theorem of elliptic functions. The literature of MALFATTI's problem is very large. See A. WITTSTEIN, *Geschichte des Malfatti'schen Problems.* Diss. Erlangen. Munich, 1871, 39 pp.+4 folding plates. Supplement, Nördlingen, 1878, 27 pp.+2 plates. Also R. C. ARCHIBALD, "Malfatti's problem," *Scripta Mathematica*, v. 1, pp. 170–171, 1932.

PAPPUS stated the theorem that of all segments of a circle whose boundaries are of equal length the half circle has the greatest area. The corresponding result for the sphere was stated by ARCHIMEDES. In 1841 STEINER gave what he believed to be complete proofs that: Of all plane figures of equal boundary length, the circle has the greatest area; and of all figures of equal area the circle has the least boundary. But the proof was faulty since he did not prove that among all closed plane curves of given length there exists one whose area is a maximum, and similarly for the sphere. F. EDLER was the first (1882) to complete STEINER's proof for the circle. The maximum property for the sphere was first rigorously proved by H. A. SCHWARZ (1884), using WEIERSTRASSIAN results in the field of calculus of variations. The best elementary discussion of the problems here mentioned is in W. BLASCHKE, *Kreis und Kugel.* Leipzig, 1916, x, 169 pp. See also M. ZACHARIAS, *Encykl. d. math. Wissen.*, v. 3_1, 1921, pp. 1118–1137; and R. STURM, *Maxima und Minima in der elementaren Geometrie.* Leipzig u. Berlin, 1910, vi, 138 pp.

277. PONCELET, *Traité des Propriétés Projectives des Figures.* Paris, 1922, pp. 187–190. Also second ed., v. 1, 1865, pp. 181–184, 413–414. In particular all points determined with ruler and compasses can also be found by a carpenter's square or a ruler with parallel edges. Since a parallelogram may be drawn with a double edged ruler we recall the problem of LAMBERT (1774), often discussed later: Given a parallelogram, construct, with ruler only, a parallel to a given line. With two carpenter squares it is possible to solve the problems of trisection of an angle or duplication of a cube. See ARCHIBALD, "Constructions with a double edged ruler," *Amer. Math. Monthly*, v. 25, 1918, pp. 357–359. What PONCELET suggested as possible with ruler alone was carried out in great detail in a famous little book of STEINER: *Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und eines festen Kreises.* Berlin, 1833, 110 pp.+2 plates. There have been Polish, two Russian, and a French (partial) translations; an English edition by MARION E. STARK, with Introduction and Notes by R. C. ARCHIBALD is in the press. Through the efforts of JACOBI the Univ. of Königsberg in 1833 conferred the degree of Ph.D. on STEINER and in 1834 through the influence of JACOBI and the brothers ALEXANDER and WILLIAM VON HUMBOLDT a new chair of geometry was founded for him at the University of Berlin and at the same time he was elected a fellow of the Prussian Academy of Sciences. If, instead of the circle in the PONCELET-STEINER constructions be substituted a given central conic and one of its foci, then every point constructed

with ruler and compasses may be found with ruler alone. Also every problem of the third and fourth degrees could then be solved with ruler and compasses. This was first proved by H. J. S. SMITH, "Mémoires sur quelques problèmes cubiques et biquadratiques," *Annali di Matem.*, s. 2, v. 3, 1869, pp. 112–165, 218–242, or *Coll. Math. Papers*, v. 2, Oxford, 1894, pp. 1–66. For this, and a paper by KORTUM, the Prussian Academy of Sciences made a joint award of its STEINER Prize for 1868.

STEINER became acquainted with the Norwegian genius ABEL when he was in Berlin, and also with A. L. CRELLE who in 1826 founded the famous mathematical *Journal f. d. reine u. angew. Mathematik*. Four of STEINER's papers and one of ABEL's appeared in the first v. of this periodical.

278. SMITH [150], "Brianchon's theorem," ed. N. A. COURT, pp. 331–336. See also [275].

279. L. WANTZEL, "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas," *Jn. de Math.* (LIOUVILLE), v. 2, 1837, pp. 366–372, particularly pp. 369 f. F. CAJORI, "Pierre Laurent Wantzel," Amer. Math. Soc., *Bull.*, v. 24, pp. 339–347, 1918.

279a. APOLLONIUS of Perga in his *Plane Loci* showed that the inverse of a circle is a circle. It is prop. 17, bk. 1, in the restoration, ROBERT SIMSON, *Apollonii Pergaei Locorum Planorum Liber II*, Glasgow, 1749; German translation by J. W. CAMERER, Leipzig, 1796. In effect the proposition is as follows: If from any two points A, B, lines $AC_1=r_1$, $BC_2=r_2$, are drawn, making a constant angle with one another, if $r_1r_2=k^2$, a constant, and if the locus of C_1 is a circle, so also is the locus of C_2, in general. For the more general proposition see PAPPUS, *Collection*, ed. by F. HULTSCH, Berlin, v. 2, 1877, pp. 661f; VER EECKE's French translation, Paris and Bruges, 1933, v. 2, p. 495.

280. PEAUCELLIER, a French army officer, announced his wonderful invention of a compass to draw a straight line, in a letter published in *Nouv. Annales de Math.*, s. 2, v. 3, 1864, p. 414. PEAUCELLIER there merely proposed as a problem to find a "compas" to describe in a continuous manner (1) a right line, (2) a circle, whatever its radius, (3) the conics. The letter concludes "Le mode de construction de ce genre du compas, supprimant tout mouvement de glissement, le tracé des courbes precitées est susceptible d'une extrême précision."

Announcement of PEAUCELLIER's invention of 1864 was made in 1867, without description, by a brother officer of engineers, A. MANNHEIM, at a meeting of the Paris Philomathic Society; but this announcement attracted no particular attention until 1871 when the mechanism was rediscovered by a young man named LIPKIN (Akad. Nauk., St. Petersburg, *Bull.*, v. 16, 1871, pp. 57–60) a pupil of the celebrated CHEBYSHEV (of the Univ. of St. Petersburg) who had been laboring to demonstrate the impossibility of that which his pupil thus achieved. LIPKIN got a substantial reward from the Russian Government for his supposed originality. Thereafter PEAUCELLIER's merit was at last recognized and he was awarded the "Prix MONTYON"—the great mechanical prize of the Institut de France. PEAUCELLIER's first paper on the linkage was published in *Nouv. Annales de Math.*, s. 2, v. 12, 1873, pp. 71–78, and linkages for conic sections, conchoids of a circle, cissoid, were described. SYLVESTER proposed (with interesting comment) the PEAUCELLIER cell as a problem in the *Educ. Times* (*Reprint*, v. 21, 1874, p. 58). PEAUCELLIER's cell had 7 links. It was HARRY HART who discovered the five-bar linkage (*Cambr. Mess. Math.*, v. 4, pp. 82–88, 116–120, 1874).

281. D. H. LEAVENS, "Linkages," *Amer. Math. Monthly*, v. 22, 1915, pp. 330–334; and C. M. HEBBERT, "A cardiograph," *idem*, pp. 12–13. A. EMCH, *An Introduction to Projective Geometry*, New York, 1905, pp. 242–260. V. LIGUINE, "Liste des travaux sur les systèmes articulés" *Bull. d. Sci. Math.*, v. 18, 1883, pp. 145–160; [151 titles published before 1883.] In *Tôhoku Math. Jn.*, v. 37, 1933, pp. 294–319, R. KANAYAMA gave a bibliography of 306 titles (1631–1931) including those of LIGUINE's list. That even this last list is not by any means exhaustive even to the end of 1931, is shown in *Scripta Math.*, v. 2, pp. 293–294, 1934.

282. CAJORI [2], pp. 349–350. The proof of RUFFINI appears in his book *Teoria generale delle Equazioni*, Bologna, 1799, and in later articles on the subject. H. BURKHARDT, "Die Anfänge der Gruppentheorie und P. R.," *Abh. z. Gesch. d. mathem. Wissen.*, Heft 6, 1892, pp. 119–159. E. BORTOLOTTI, *Influenza dell'Opera matematica di Paolo Ruffini sullo svolgimento delle teorie algebriche*. Modena, 1902, 57 pp. See also [215].

283. W. A. J. ARCHBOLD, "W. G. Horner," Dict. Nat. Biog., v. 27, 1891; SMITH [150], "Horner's method," ed. by M. McGUIRE, pp. 232–252. HORNER'S first paper on the subject was publ. in R. Soc. London, *Trans.*, v. 109, 1819, pp. 308–335. It was reprinted, with some additional matter in *Ladies Diary*, 1838, pp. 49–72. Two revisions appeared in LEYBOURN'S *Math. Repository*, v. 5, 1830, part II, pp. 21–75, and in *The Mathematician*, "On algebraic transformations," v. 1, 1845, pp. 108–116, 136–142, 311–316. F. CAJORI, "Horner's method of approximation anticipated by Ruffini," Amer. Math. Soc., *Bull.*, v. 17, 409–414, 1911. See also [215].

284. CAJORI [2], pp. 363–364. WOLF [222], v. 4, 1862, pp. 375–389. M. BÔCHER, "The published and unpublished work of Charles Sturm on algebraic and differential equations," Amer. Math. Soc., *Bull.*, v. 18, pp. 1–18, 1911. G. LORIA, "Charles Sturm et son oeuvre mathématique," *L'Enseign. Math.*, v. 37, 1939, pp. 249–272+portrait plate [246].

285. If the roots of the equation $f(x)=0$ are under consideration the theorem states that if N is the number of roots in the complex plane within a closed circuit A, not passing through any of the roots, then

$$N = \frac{1}{2\pi i}\int \frac{f'(x)}{f(x)}\,dx,$$

the integral being taken around A. More generally, if $f(x)$ is continuous except for a finite number of poles, N is the difference between the number of poles and the number of zeros of $f(x)$. See CAUCHY, *Bull. Sci. Math.*, v. 16, 1831, pp. 116–128; Paris, École Polytechnique, *Jn.*, cahier 25, v. 15, 1837, pp. 176–193; C. STURM & J. LIOUVILLE. "Démonstration d'un théorème de M. Cauchy, relatif aux racines imaginaires des équations," *Jn. d. Math.* (LIOUVILLE), v. 1, 1836, pp. 278–289, and C. STURM, "Autres démonstrations du même théorème," *idem*, pp. 290–308. A reference may also be given to E. MCCLINTOCK, "A method for calculating simultaneously all the roots of an equation," *Amer. Jn. Math.*, v. 17, 1895, pp. 89–110.

286. BELL [53], "Mathematics and windmills—Cauchy," pp. 294–306. PRASAD [254], v. 1, pp. 68–110. SMITH [54]. CAJORI [2], pp. 368–370. R. STRUIK & D. J. STRUIK, "Cauchy and Bolzano in Prague," *Isis*, v. 11, pp. 364–366, 1928. J. BERTRAND, *Éloges Académiques*, Nouv. série. Paris, 1902, "Éloge de Augustin Louis Cauchy," pp. 101–120. P. E. B. JOURDAIN, "The origin of Cauchy's conceptions of a definite integral and of the continuity of a function," *Isis*, v. 1, pp. 661–703, 1914. SMITH [150], "Möbius, Cayley, Cauchy, Sylvester, and Clifford on geometry of four or more dimensions," ed. by H. P. MANNING, p. 524; "Cauchy on higher space" ed. by H. P. MANNING, pp. 530–531; "Cauchy on the derivatives and differentials of functions of a single variable," ed. by E. WALKER, pp. 635–637. I. TODHUNTER, *A History of the Progress of the Calculus of Variations in the Nineteenth Century*. Cambridge and London, 1861, pp. 210–228. etc. C. A. VALSON, *La Vie et les Travaux du Baron Cauchy*. Paris, 2 v., 1868, xxiv, 290, xxiv, 178 pp. CAUCHY, *Oeuvres Complètes*. 25 v. Paris, 1882–1938, to the end of 1947.

287. A. SACHSE, "Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variablen durch trigonometrische Reihen," *Abh. z. Gesch. d. Math.*, Heft 3, 1880, pp. 229–276. BÔCHER [246]. LANGER [246].

288. Herewith is a selected list of references on this topic, also indicating its history: HILDA HUDSON, *Ruler and Compasses*, London, 1916, pp. 112–117. J. S. MACKAY, "Geometrography of Euclid's problems," Edinb. Math. Soc., *Proc.*, v. 12, 1894, pp. 2–16. J. L. COOLIDGE, *A Treatise on the Circle and the Sphere*. Oxford, 1916, pp. 166–179; applications to the problem of APOLLONIUS and MALFATTI'S problem. É. LEMOINE, "De la mésure de la simplicité dans les constructions géométriques," Acad. d. Sci., Paris, *Comptes Rendus*, v. 107, 1888, pp. 169–171. É. LEMOINE, *Géométrographie ou Art des Constructions Géométriques*. (*Scientia* series, no. 18). Paris, 1902, 87 pp. ROUCHÉ & COMBEROUSSE, *Traité de Géométrie*. Paris, seventh ed., 1900; Note IV: "Sur la géométrographie," by E. L., pp. 517–548. A. ADLER, *Theorie der geometrischen Konstruktionen*. Leipzig, 1906, pp. 277–301. T. VAHLEN, *Konstruktionen und Approximationen in systematischer Darstellung*. Leipzig 1911; "Geometrographie und Fehlertheorie," pp. 121–128. E. PAPPERITZ, "Darstellende Geometrie," *Encykl. d. math. Wissen.*, v. 3_1, part 4, 1910, pp. 529–531, 536. C. ALASIA, *La recente Geometria del Triangolo*. Citta di Castello, 1900, xvi, 339 pp.; "La geometrografia," pp. 29–49, 282–

283. A different system from that of Lemoine is set forth in A. GRÜTTNER, *Die Grundlagen der Geometrographie.* Progr. Breslau. Leipzig, 1912, 54 pp. On pp. 7–8 GRÜTTNER remarks that already in 1833, in his work on *Geometrical Constructions* [277], STEINER formulated the thought which is the basis of geometrography. LEMOINE wrote also on geometrography of three dimensions, *Mathesis*, 1902, pp. 105–107. LEMOINE was really the founder of *L'Intermédiaire d. Mathématiciens*, of which he and his friend C. A. LAISANT were the editors, v. 1–17, 1894–1910. For sketches, each with a portrait of LEMOINE, see D. E. SMITH, *Amer. Math. Monthly*, v. 3, 1896, pp. 29–33, and C. A. LAISANT, *L'Enseignement Math.*, v. 14, 1912, pp. 177–183. Then the v. by L. AUGÉ DE LASSUS, *La Trompette un demisiècle de musique de Chambre*, Paris, 1911, ii, 239 pp., tells of the remarkable half-century of gatherings at LEMOINE's home of a good part of the scientific, literary, and artistic circles of Paris, for the celebrated musical soirées.

289. For example, there are *Points* associated with such names as: BROCARD, CRELLE, GAUSS, JEŘÁBEK, LEMOINE, NAGEL, STEINER, TARRY; among *Lines:* antiparallel, BROCARD, isogonal, LEMOINE, PHILO, symmedian; among *Triangles:* BROCARD, FUHRMANN, KIEPERT, LIONNET, MORLEY; among *Circles:* APOLLONIUS, JOACHIMSTHAL, LONGCHAMPS, M'CAY, SCHOUTE, SPIEKER, TAYLOR, TUCKER; among *Ellipses:* BROCARD, CESÀRO, STEINER; among *Hyperbolas:* APOLLONIUS, FEUERBACH, KIEPERT; among *Parabolas:* ARTZ, KIEPERT, MANDART, NEUBERG; and among *Theorems:* CATALAN, CEVA, CHAPPLE, LUCAS, MENELAUS, SCHLÖMILCH.

A few selected references in this field are as follows: CAJORI [3], pp. 259–263. C. ALASIA, *La Recente Geometria del Triangolo.* Città di Castello, 1900, xvi, 339 pp.; there are 566 formulae for the triangle, pp. 309–339. C. ALASIA, *Saggio Terminologico-Bibliografico sulla Recente Geometria del Triangolo.* Bergamo, 1902, iv, 43 pp. J. CASEY, (a) *A Sequel to the First Six Books of the Elements of Euclid.* Fifth edition rev. and enl. Dublin and London, 1888, pp. 165–248; not in earlier or later English eds.—French ed. Ghent and Paris, 1890, 80 pp. of the pages 165–248, and of an unpubl. supplement by CASEY. (b) *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections.* Second ed. rev. and enl., Dublin and London, 1893, pp. 418–461. J. S. MACKAY, (a) History of the nine point circle," pp. 19–57; (b) "Early history of the symmedian point, pp. 92–103, Edinb. Math. Soc., *Proc.*, v. 11, 1893. A. EMMERICH, *Die Brocardschen Gebilde und ihre Beziehungen zu den verwandten merkwürdigen Punkten und Kreisen des Dreiecks.* Berlin, 1891, 154 pp.+1 folding plate. R. A. JOHNSON, *Modern Geometry of the Triangle and the Circle.* Boston, 1929. xiv, 319 pp. W. J. M'CLELLAND, *A Treatise on the Geometry of the Circle. . . .* London, 1891, pp. 60–120, 207–217, etc. J. J. MILNE, *Companion to the Weekly Problem Papers.* London, 1888, pp. 99–191, by T. C. SIMMONS. R. C. J. NIXON, *Euclid Revised.* Third ed. Oxford, 1899; "The modern geometry of the triangle," pp. 378–403.

290. E. B. WILSON, "Reminiscences of Gibbs by a student and colleague," Amer. Math. Soc., *Bull.*, v. 37, pp. 401–416, 1931. R. E. LANGER, "Josiah Willard Gibbs," *Amer. Math. Monthly*, v. 46, 1939, pp. 75–84. E. B. WILSON, *Dict. Amer. Biog.*, "Gibbs," v. 7, 1931, pp. 248–251. H. A. BUMSTEAD, biographical sketch and bibliography in (a) *Amer. Jn. Sci.*, s. 4, v. 16, 1903, pp. 187–202; (b) *The Collected Works of J. Willard Gibbs*, 2 v., v. 1, New York, 1928, pp. xiii–xxviii. *A Commentary on the Scientific Writings of J. Willard Gibbs.* Edited by F. G. DONNAN and A. HAAS. 2 v., New Haven, 1936; each of these v. deals with the corresponding v. of the *Collected Works.* MURIEL RUKEYSER, *Willard Gibbs*, New York, 1942, xi, 465 pp.; an interesting (though far from competent) biography written by a poet. In 1923 the American Mathematical Society established the JOSIAH WILLARD GIBBS [honorary] Lectureship, for annual semi-popular lectures on various aspects of mathematics and its applications.

291. H. APPLEYARD, *Dict. Nat. Biog. 1922–1930*, 1937. A. RUSSELL, "Mr. Oliver Heaviside, F. R. S.," *Nature*, v. 115, 1925, pp. 237–238. E. T. WHITTAKER, "Oliver Heaviside," Calcutta Math. Soc., *Bull.*, v. 20, 1930, pp. 199–220.

292. At three years of age HAMILTON was a superior reader of English and considerably advanced in arithmetic; at four a good geographer; at five able to read and translate Latin, Greek and Hebrew and liked to recite DRYDEN, COLLINS, MILTON, and HOMER; at eight a reader of Italian and French and giving vent to his feelings in extemporized Latin; at ten a student of Arabic and Sanscrit. His career at Trinity College, Dublin, was extraordinary in that he achieved the

previously unheard of distinction of winning the highest possible marks, both in mathematical physics and in Greek. *A Collection of Papers in memory of Sir William Rowan Hamilton* (*Scripta Mathematica Studies*, no. 2). New York, 1945, 82 pp.+2 plates. [Partial contents: Facsimile pages from HAMILTON's Notebooks, pp. 12, 36. J. L. SYNGE, "The life and early work of W. R. H.," pp. 13–24. C. C. MACDUFFEE, "Algebra's debt to Hamilton," pp. 25–35; F. D. MURNAGHAN, "An elementary presentation of the theory of quaternions," pp. 37–49; "Poems by William Rowan Hamilton," p. 50. H. BATEMAN, "Hamilton's work in dynamics and its influence on modern thought," pp. 51–63. "The Hamilton postage stamp" pp. 81–82.] A. MACFARLANE, *Lectures on Ten British Mathematicians of the Nineteenth Century*. New York, 1916, 148 pp. W. R. H., pp. 34–39; [the other sketches are of G. PEACOCK, A. DEMORGAN, G. BOOLE, A. CAYLEY, W. K. CLIFFORD, H. J. S. SMITH, J. J. SYLVESTER, T. P. KIRKMAN, I. TODHUNTER]. SMITH [54]. BELL [53], "An Irish tragedy—Hamilton," pp. 340–361. "Quaternion centenary celebration [8 Nov. 1943]," R. Irish Acad., *Proc.*, v. 50A., pp. 69–122+2 plates, 1945. [Partial contents: Opening remarks by the president; messages from J. L. SYNGE and G. D. BIRKHOFF; "The Dublin mathematical school in the first half of the nineteenth century" by A. J. MCCONNELL, pp. 75–88; "Quaternions" HAMILTON ms. 16 Oct., 1843, pp. 89–90; "The sequence of ideas in the discovery of quaternions," pp. 93–98.] H. T. H. PIAGGIO, "The significance and development of Hamilton's quaternions," *Nature*, v. 152, 1943, pp. 553–555. R. P. GRAVES, *Life of Sir William Rowan Hamilton . . . including Selections from his Poems, Correspondence and Miscellaneous Writings*. Dublin, v. 1, 1882, 698 pp.+plate; v. 2, 1885, xvi, 719 pp.+plates; v. 3, 1889, xxxvi, 673 pp.+plates; Addendum, 1891, 17 pp. with particular reference to corrections of R. E. ANDERSON, *Dict. Nat. Biog.*, W. R. H., v. 24, 1890. R. C. ARCHIBALD, "Mathematicians, and poetry and drama," *Science*, v. 89, 1939, pp. 19–26, 46–50; W. R. H., pp. 25–26. Two v. of *The Mathematical Papers of Sir William Rowan Hamilton* have been published: v. 1, *Geometrical Optics*, ed. by A. W. CONWAY & J. L. SYNGE. Cambridge, 1931, xxx, 534 pp.; Éloge by CHARLES GRAVES, pp. xvii–xxviii; v. 2, *Dynamics*, ed. by A. W. CONWAY & A. J. MCCONNELL, 1940, xviii, 656 pp.

Having referred to GIBBS, HEAVISIDE, and HAMILTON, I should also, perhaps, have mentioned a German genius, HERMANN GÜNTHER GRASSMANN (1809–77) and his two "Ausdehnungslehren" (1844, 1862), forming the basis of vector analysis. The first work in English on this subject was *The Directional Calculus Based upon the Methods of Hermann Grassmann* by E. W. Hyde, Boston, 1890, xii, 247 pp. SMITH [150], "Grassmann on the Ausdehnungslehre" ed. by M. KORMES, pp. 684–696. Commenting on GRASSMANN's many-sided make-up and enormous industry (Amer. Math. Soc., *Bull.*, v. 22, 1916, pp. 150), E. B. WILSON noted that he started as a theologian, invented his analysis of extension, wrote on physics, composed class texts for the study of German, Latin, and mathematics, edited a political paper and a missionary paper, investigated phonetic laws, published a dictionary of the Rig-Veda and a translation of it into verse, harmonized folk-songs in three voices—in addition to carrying on successfully his regular work as a teacher and an administrator, and to bringing up nine of his eleven children.

293. *Mathematical Tables and Other Aids to Computation*, nos. 1–24 (Jan. 1943–Oct. 1948).

294. D. H. LEHMER, Washington, D. C., 1941, xiv, 177 pp.

295. R. C. ARCHIBALD, "The New York Mathematical Tables Project," *Science*, n. s., v. 96, 25 Sept. 1942, pp. 294–296.

296. J. W. L. GLAISHER, 175 pp. See also GLAISHER's articles in *Encycl. Britannica*, eleventh ed., 1911, art. "Logarithm," v. 16, pp. 868–877, and "Table mathematical," v. 26, pp. 325–336. There are also valuable older articles of A. DE MORGAN, in (a) *Penny Cycl.*, art. "Table," v. 23, 1842, pp. 496–501; and *Suppl.*, v. 2, 1846, pp. 595–605. (b) *Engl. Cycl., Arts and Sci. Sect.*, v. 7, 1861, cols. 976–1016. See also reports of the Tables' Comm. in BAAS, *Report* 1875, 32 pp.; 1878, 6 pp.+1 pl.; 1879, 12 pp.+1 pl.; 1880, 9 pp.+1 pl.; 1881, 6 pp.; 1883, 8 pp. See also annual Committee reports 1930–1939; and the history, 1871–1948, *MTAC*, Jan. 1949.

297. JAMES HENDERSON, *Bibliotheca Tabularum Mathematicarum being a Descriptive Catalogue of Mathematical Tables. Part I, Logarithmic Tables, A. Logarithms of Numbers.* (*Tracts for Computers*, no. XIII.) Cambridge, 1926, viii, 208 pp. It is here indicated that W. W. DUFFIELD, superintendent of the U. S. Coast and Geodetic Survey, was guilty in 1896 of publishing a 10D table of numbers, mostly taken from VEGA's fine tables of a century before, *Thesaurus Logarithmorum*

Completus. Leipzig, 1794. See also *Math. Tables and Other Aids to Computation*, v. 2, pp. 161-165, 311-312.

298. A. FLETCHER, J. C. P. MILLER & L. ROSENHEAD, London, 1946, viii, 451 pp.

299. H. ANDOYER, *Nouvelles Tables Trigonométriques Fondamentales* (Valeurs Naturelles). 3 v., Paris, 1915–1918, xviii, 341, iv, 275, iv, 367 pp.

300. D. N. LEHMER, (a) *Factor Table for the First Ten Millions. . . .* Washington, 1909, 488 pp.; (b) *List of Prime Numbers from* 1 to 10,006,721. Washington, 1914, 143 pp.

301. J. C. BURCKHARDT, (a) *Table des Diviseurs pour tous les Nombres des 1er, 2e, et 3e Million. Ou plus exactement, depuis 1 à 3 036000.* Paris, 1817, xii, 114, iv, 112, iv, 112 pp. There was a facsimile edition of this v. published at Berlin in 1909.

302. J. GLAISHER (a) *Factor Table for the Fourth Million*, London, 1879, 52 [112] pp.+3 plates; (b) *Factor Table for the Fifth Million*, 1880, 14, (112) pp.; (c) *Factor Table for the Sixth Million*, 1883, 104, (112) pp.

302a. A. J. C. CUNNINGHAM, *A Binary Canon.* 1900, viii, 172 pp. See ARCHIBALD [**163**], pp. 18–21, and *Math. Tables and Other Aids to Computation*, v. 1, p. 109; v. 3, pp. 143–144.

303. L. WOODRUFF, ed., *The Development of the Sciences.* New Haven, 1923; Mathematics by E. W. BROWN, pp. 1–42; Physics by H. A. BUMSTEAD, pp. 43–73; Astronomy by F. SCHLESINGER, pp. 126–167. Second series, New Haven, 1941; lectures on the history of science organized by the Yale chapter of the Gamma Alpha Graduate Scientific Fraternity. [Mathematics by O. ORE, pp. 1–51; Physics by H. MARGENAU, pp. 91–120; Astronomy by F. SCHLESINGER, pp. 53–89.]

304. F. KLEIN, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, 2 v., Berlin, 1926–1927, xiv, 386, x, 208 pp.

305. R. COURANT, "Riemann und die Mathematik der letzten hundert Jahre," *Naturwissen.*, v. 14, 1926, pp. 813–818.

306. R. S. WOODWARD, "The century's progress in applied mathematics," Amer. Math. Soc., *Bull.*, v. 6, pp. 133–163, 1900; retiring presidential address, 1899.

307. New York, 1938, viii, 315 pp. [Contents: E. T. BELL, "Fifty years of algebra in America," pp. 1–34; J. F. RITT, "Algebraic aspects of the theory of differential equations," pp. 35–55; N. WIENER, "The historical background of harmonic analysis," pp. 56–68; E. J. MCSHANE, "Recent developments in the calculus of variations," pp. 69–97; T. Y. THOMAS, "Recent trends in geometry," pp. 98–135; R. L. WILDER, "The sphere in topology," pp. 136–184; G. C. EVANS, "Dirichlet problems," pp. 185–226; J. L. SYNGE, "Hydrodynamical stability," pp. 227–269; G. D. BIRKHOFF, "Fifty years of American mathematics," pp. 270–315.]

308. R. C. ARCHIBALD, *A Semicentennial History of the American Mathematical Society 1888–1938 with Biographies and Bibliographies of the Past Presidents.* New York, 1938, pp. 107–244.

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313. PROCLUS DE LYCIE, *Les Commentaires sur le premier Livre des Éléments d'Euclide. Traduite pour la première fois du Grec en Français, avec une Introduction et des Notes*, by P. VER EECKE. Bruges, 1948, xxiv, 372 p. *The Philosophical and Mathematical Commentaries of Proclus on the First Book of Euclid's Elements. . . .* Translated by THOMAS TAYLOR. 2 v., London, 1792, cxxxii, 183, iv, 444 p. There was a German edition by MAX STECK & L. SCHÖNBERGER, Halle, 1945.

314. *The Pneumatics of Hero of Alexandria from the original Greek.* Translated and edited by BENNET WOODCROFT. London, 1851, xx, 118 p. A. G. DRACHMANN, *Ktesibios, Philon and Heron. A Study in Ancient Pneumatics. Publications of the University Library*, Copenhagen, v. 4; *Scientific and Medical Department.* Copenhagen, 1948, xii, 197 p. Water-clocks are discussed, p. 16–41; see [**39**].

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"Thou hast made me known to friends whom I knew not. Thou hast given me seats in homes not my own. Thou hast brought the distant near and made me a brother of the stranger."

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